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Monte Carlo Matrix Calculation with Punched Card Machines

FORSYTHE & LIEBLER have described matrix inversion by a Monte Carlo method [*MTAC*, v. 4, p. 127]. While they state that the method is "best suited for a human computer with a table of random digits and no calculating machine," the principle has been adapted for use with punched card machines. The method has been extended to finding powers of a matrix. While an IBM 405 tabulator was used in this work, there is no reason why a newer machine cannot be used.

The class of matrices $B = (b_{ij})$ that may be inverted by this method is limited to those in which

(1) the largest latent root of $A^* = (a_{ij}^*)$ is less than unity, where a_{ij}^* is the absolute value of $a_{ij} = \delta_{ij} - b_{ij}$ and

(2) the sum of the elements in any row of A^* is less than unity

In certain cases these restrictions may be removed.

In this modification of the method of Forsythe and Liebler, the illustrative urns containing numbered balls are replaced by a group of punched cards. Each card is divided into two principal fields: a randomizing field consisting of 30 to 50 random digits and a playing field prepared for each matrix as described below. Each of the n card columns of the playing field of the deck corresponds to one of the urns. Each of the n different digits punched in the card column corresponds to the numbered balls in that urn. An x-punch in a card column corresponds to a "stop ball" in the urns of the original description. The deck of cards is treated as a unit and is so prepared that the probability of a j appearing in the ith card column is p_{ij}. In the case where A has both positive and negative elements, a value of -1 is attached to each draw corresponding to a negative element, $+1$ to all others. The negative value is indicated by a 12 punch appearing in the corresponding card column above the row punch.

In terms of the original reference, n games are played simultaneously to yield one row of the desired inverse B^{-1}. The passage of each card through the machine constitutes one draw in the game.[1] The machine is wired so that the original gambling procedure is followed. Instructed by a digit emitter, the machine reads the card column corresponding to the ith row. The digit punched in that column activates the mechanism which selects one column of the next card for reading. The digit in each column selected causes the reading of the corresponding column in the following card. This procedure is continued until an x is read. The column from which the x was read scores ± 1, all others score zero. As the values are to be multiplied together, the score is -1 if an odd number of negative draws was made, otherwise the score is $+1$. A designated counter keeps count of the total games played. The machine does not stop after each game, but starts anew on the card following the x card, again instructed by the digit emitter. After the entire deck of cards passes through, a total is taken and the machine prints total draws, total games and net score for each of the n

games. The actual elements of the inverse matrix are obtained by dividing by the product of the stop probability times the number of games played. If the stop probability can be taken as 0.1, and the number of games as an exact power of 10, the machine will actually list the elements found. By playing successively, starting with $i = 1$ and continuing through $i = n$, the entire inverse will be listed with the elements in their correct positions.

By a simple modification, powers of certain matrices may be found by playing a similar Monte Carlo game. In this case, the stop probabilities are no longer used to terminate each game. Instead, exactly $r + 1$ draws are made and the scoring is performed as before. As all possible routes from i to j consisting of exactly r steps will be covered in a large number of games, this is tantamount to raising the matrix to the rth power. The restriction on the size of the largest latent root no longer applies here, but it is required that $\sum_{j=1}^{n} a_{ij} = 1$. The instruction to score every $(r + 1)$st draw may be given the machine by interspersing control cards by means of a counting collator or by sequencing using interlocked selector relays as described below.

The order of matrices that may be raised to powers or inverted depends on the selector capacity of the machine. For general functions, one single-position selector, one three-position selector and one ten-position selector are required as well as two digit-selectors. For matrices of the nth order, n two-position selectors and n single-position selectors are also needed.

The advantages inherent in this method are:

1. It may be used with a common machine present in nearly all punched card equipped installations.
2. The actual working procedure is far simpler than for any previously reported matrix inversion method.
3. Approximate inverses may be obtained rapidly. The accuracy of a good approximation to an inverse may be easily improved.[2]
4. Using a machine with large selector capacity,[3] approximate inverses and powers may be obtained for high order matrices. In going from a matrix of the nth order to one of the $(2n)$th order, twice as many cards should go through the machine in twice the number of passes. Thus, as originally stated, the number of operations is proportional to the second instead of the third power of the order, as is otherwise the case.
5. The process of finding all powers from 2 to r is an exceedingly simple one. It should be useful in evaluation of latent roots and vectors.

Example 1—Inversion of a seventh order symmetrical matrix

$$B = \begin{vmatrix} .7 & -.1 & -.1 & -.1 & -.1 & -.1 & -.1 \\ -.1 & .7 & -.1 & -.1 & -.1 & -.1 & -.1 \\ -.1 & -.1 & .7 & -.1 & -.1 & -.1 & -.1 \\ -.1 & -.1 & -.1 & .7 & -.1 & -.1 & -.1 \\ -.1 & -.1 & -.1 & -.1 & .7 & -.1 & -.1 \\ -.1 & -.1 & -.1 & -.1 & -.1 & .7 & -.1 \\ -.1 & -.1 & -.1 & -.1 & -.1 & -.1 & .7 \end{vmatrix}$$

$$A = \begin{vmatrix} .3 & .1 & .1 & .1 & .1 & .1 & .1 \\ .1 & .3 & .1 & .1 & .1 & .1 & .1 \\ .1 & .1 & .3 & .1 & .1 & .1 & .1 \end{vmatrix} \qquad p_j = \begin{vmatrix} .1 \\ .1 \\ .1 \end{vmatrix}$$

etc. etc.

$$B^{-1} = |2.67 \quad 1.33 \quad 1.66 \quad 1.33 \quad 1.55 \quad 1.00 \quad 1.33|$$

etc.

(90 games per element)

$$B^{-1} = |2.536 \quad 1.224 \quad 1.288 \quad 1.275 \quad 1.263 \quad 1.263 \quad 1.263|$$

etc.

(784 games per element)

$$B^{-1} = |2.500 \quad 1.250 \quad 1.250 \quad 1.250 \quad 1.250 \quad 1.250 \quad 1.250|$$

etc.

(calculated values)

Example 2—A seventh order symmetrical matrix raised to the seventh power.

$$A = \begin{vmatrix} .88 & .02 & .02 & .02 & .02 & .02 & .02 \\ .02 & .88 & .02 & .02 & .02 & .02 & .02 \\ .02 & .02 & .88 & .02 & .02 & .02 & .02 \\ .02 & .02 & .02 & .88 & .02 & .02 & .02 \\ .02 & .02 & .02 & .02 & .88 & .02 & .02 \\ .02 & .02 & .02 & .02 & .02 & .88 & .02 \\ .02 & .02 & .02 & .02 & .02 & .02 & .88 \end{vmatrix}$$

$$A^7 = \begin{vmatrix} .440 & .120 & .088 & .072 & .080 & .088 & .112 \\ .121 & .427 & .096 & .081 & .081 & .096 & .096 \\ .111 & .079 & .428 & .103 & .095 & .087 & .095 \end{vmatrix}$$

etc.

(125 games per element)

$$A^7 = |.4388 \quad .0933 \quad .0954 \quad .0952 \quad .0946 \quad .0946 \quad .0946|$$

etc.

(825 games per element)

$$A^7 = |.4410 \quad .0932 \quad .0932 \quad .0932 \quad .0932 \quad .0932 \quad .0932|$$

etc.

(calculated values)

Example 3—A seventh order nonsymmetrical matrix raised to the sixth power.

$$A = \begin{vmatrix} .88 & .02 & .02 & .02 & .02 & -.02 & .02 \\ .02 & .88 & .02 & .02 & .02 & -.02 & .02 \\ .02 & .02 & .88 & .02 & .02 & -.02 & .02 \\ .02 & .02 & .02 & .88 & .02 & -.02 & .02 \\ .02 & .02 & .02 & .02 & .88 & -.02 & .02 \\ .02 & .02 & .02 & .02 & .02 & -.02 & .88 \\ -.02 & -.02 & -.02 & -.02 & -.02 & .88 & -.02 \end{vmatrix}$$

$$A^4 = \begin{vmatrix} .4683 & .0739 & .0880 & .0704 & .0704 & -.0422 & -.0523 \end{vmatrix}$$

(568 games per element)

$$A^4 = \begin{vmatrix} .4782 & .0736 & .0736 & .0736 & .0736 & -.0025 & -.0025 \\ .0736 & .4782 & .0736 & .0736 & .0736 & -.0025 & -.0025 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ -.0025 & -.0025 & -.0025 & -.0025 & -.0025 & -.0025 & .4782 \end{vmatrix}$$

(calculated values)

Details of the procedure.—*Card Preparation.*

A. Random numbers may be keypunched from standard tables[4] or pseudo-random numbers may be produced [*MTAC*, v. 5, p. 4].

B. Preparation of a group of cards correctly punched in the playing field may be accomplished as follows.

1. Prepare the needed table of probabilities by listing the absolute values of the matrix with rows and columns transposed. (To prepare cards for matrix inversion, include the stop probabilities as the final row in the table.) The probabilities in each column (including stop probabilities if present) should add to unity.

2. An effective, but tedious, procedure for preparing the playing field for a single card column would be to partition the deck by actual count into n (or $n + 1$ in the case of inversion) groups each containing $p_{ij}\%$ of the card count. For each group the appropriate value of j would be gang punched into the card column representing i. A 12-punch is included when a value of -1 is attached to the probability and an x-punch represents the stop cards.

3. This may also be done by preparing a table of accumulated probabilities, changing this to a table of turning points and collating with cards sorted on a d digit random field when the probabilities are given to d significant figures. The values of j as well as any x or 12 punches are interspersed gang-punched onto the playing field of the random number cards.

Example of Preparation of Table of Turning Points for One Column in the Inversion of a Fourth Order Matrix

j	*Column of Probabilities*	*Accumulated Probabilities*	*Turning Points*	j
1	.323	.323	000	1
2	−.076	.399	.323	2*
3	.458	.857	.399	3
4	.109	.966	.857	4
p_j	.034	1.000	.966	x
				Keypunch

* (also 12 punch)

4. Using either of these procedures, the cards should be sorted on several random fields between the preparation of successive columns of the playing field.

Performing the Calculation

Place the designated wire in position 1 of digit selector 2. Pass entire deck of cards through the tabulator. Press hand key for total. Successively pass cards through with wire in digit selector positions 2 to n.

Machine Wiring

1. *Counters.* Use $n + 2$ counter groups. Read card count into all groups. Add and subtract hubs go to selector F (NX and X, respectively), from C of that selector, through the delayed selectors and the entry selector to "plug to C." Net balance and clear all counters on final total, emitting an asterisk next to negative totals. One counter group is for card count; another counts games played (plug to C through entry selector).

2. *Reading Selectors.* n two-position selectors are used. Each X goes to its corresponding upper brush position in the playing field. C is common and leads to the entry selector. The X in the other position goes to the D–PU of the corresponding delayed selector. The C is again common and leads to a digit of DS 2. (UCI goes to DSC 2, making DS 2 a digit emitter.) The PU of each selector is through the corresponding hub of DS 1.

3. *Delayed Selectors.* These are picked up by a digit impulse passing into the C of all reading selectors and through the X positions to the D–PU of the corresponding selector. Thus, delayed selector 3 picks up one cycle after reading selector 3 picks up. (This is done so that the machine can store the number of the "winning" digit.) The delayed selectors carry a common "plug to C" on their C hubs: the X hubs lead to the corresponding C positions in selector F.

4. *Entry Selector.* Three-positions picked up by the first card or by an 11 from Digit Selector 1. One position carries a "plug to C" impulse to the counter which records the number of games played and to the C of the delayed selectors. A second is used as a drop-out for the negative impulse counter. The third selects either the output of the reading selector bank-NX or an emitted digit from DS 2 (which is impulsed by UCI)-X to DSC 1.

5. *Negative Impulse Counter.* To distinguish an odd from an even number of 12 impulses, the selector picks up on the first negative impulse, drops out on the second, picks up on the third, etc. An 11 emitted by DS 2 passes through the entry selector (C to NX), through a special drop-out selector (C to NX), through one position of Selector F (C to X) to the X–PU of Selector F. The 12 position in DS 1 is wired through Selector F (C to N) to the D–PU of that selector. The corresponding X position carries the 12 impulse to the special drop-out selector. By this means, Selector F remains picked up after odd 12's, and drops out on all even 12's and as each game starts.

6. *Sequencing Circuit.* By passing an x through one distributor (C to X) to the X–PU of the next distributor, a chain of $r + 1$ distributors may be linked together. If the $(r + 1)$st distributor picks up the entry selector and

starts the chain over again, every $(r + 1)$st draw will be scored. This circuit is used for raising matrices to powers without employing control cards.

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[1] This is actually sampling without replacement and therefore does not strictly conform to the MARKOV process originally described. However, if the ratio of the number of punched cards to the order of the matrix is very large, nonreplacement has negligible effect on the remaining transition probabilities until the exhaustion of the deck approaches.

[2] H. HOTELLING, "Some new methods in matrix calculation," *Ann. Math. Stat.* v. 14, 1943, p. 1–34.

[3] The IBM 101 statistical machine should be able to invert matrices up to the 30th order because of its selector capacity.

[4] Random numbers on punched cards are available from the RAND Corp., Santa Monica, Calif.

Computing Logical Truth with the California Digital Computer

1. Introduction. A problem which occasionally arises and in quite surprisingly diverse fields of endeavor is the computation of a truth table for a sentence built up of simple sentences connected by the simple sentential connectives. The problem is to find the truth value of the complete sentence for all combinations of truth values for the component sentences. If the complete sentence is a combination of n two-valued sentences, then it itself is two-valued and 2^n possible conditions must be considered. Thus if the number of different component sentences is at all large, an extremely tedious computation is necessary.

Since the computation itself is performed in a routine manner, the possibility of using a high-speed, automatic computer to carry through the details suggests itself. Indeed, a machine for this purpose (the Kalin-Burkhart Logical-Truth Calculator) has already been built.[1] It seems desirable, however, to investigate the possibility of solving these problems using general-purpose, digital computers as many of these will be in operation within the next few years. With this in mind, a program for truth-table calculation is worked out in the following for the California Digital Computer [MTAC, v. 5, p. 57–61]. The general plan should be applicable to any digital computer.

Most problem-solving programs require three basic routines. First the solution of the problem with a given set of data, second storing the result at the proper location in the memory, and third altering the data as required for the next computation. These will be discussed in subsequent sections.

Since the details of a computation program depend to a considerable extent on the characteristics of the computer for which it is developed, a brief description of the operations needed and available in the California Digital Computer (commonly called Caldic) will be presented. Other operations than those described are also available, but the discussion will not be complicated by describing them.

The program and data are given to the machine in the form of holes punched in a standard teletype tape. Results are printed out of the machine on a similar tape. Numerical data and results are in the form of decimal numbers which are used by the machine in a binary-coded form. The fact

that numbers available to the operator are decimal is of importance in programming problems. This is particularly true of problems of the sort considered here which are essentially binary in nature. The numbers contain ten decimal digits and algebraic sign.

Before computation starts, data and orders are all transferred from the tape to the internal memory of the machine. This transfer takes approximately 18 milliseconds per number. The internal memory is a magnetic drum with a capacity of 10000, ten digit numbers. Each memory position is identified by a four digit, decimal number. No differentiation is made between orders and numbers as far as the memory is concerned. The number stored in a memory box is not altered by any operation except that of putting another number in the box.

The first two digits in an order identify the operation to be performed. (The operator sees and uses them as letters, the machine as numbers.) The next four digits usually are the address of the operand in the memory. Exceptions will be discussed with the particular operations concerned. The final four digits are zeros.

The results of computations appear in the A register. It takes a special order to transfer them to the memory if this is necessary before another operation is to be performed.

The other register of interest is the order counter. The last entry on the input tape is a four digit number which is stored in the order counter and is the address of the first order. Thereafter as each order is completed the number in the order counter is increased by one to give the address of the next order. An exception is when a special order is used to change the number in the order counter.

The operations of interest are described in the following. The time for the operation should be understood to be the average time for a large number of random repetitions of the operation.

1. ad M. Average time 17.6 milliseconds. M is a four digit number. Add the number in the memory box M to the number in the A register. This operation takes proper account of the algebraic signs of the two numbers to be added. In case the operation causes the A register to overflow, the machine stops unless the next order is the cp order. (See below.) At the completion of the ad operation the sum stands in the A register but without the overflowing digit if such exists.

2. su M. Average time 17.6 milliseconds. Subtract the number in the memory box M from the number in the A register. At the end of the operation the difference stands in the A register. The remarks about sign and overflow under ad apply here also.

3. mh M. Average time 32.4 milliseconds. Multiply the number in memory box M by the number in the A register. This operation takes proper account of the algebraic sign of the two factors. It proceeds as if the decimal point is just to the left of each of the two factors. Therefore no overflow can occur.

4. tm M. Average time 17.3 milliseconds. Transfer the number in the A register to the memory box M. This operation leaves the A register clear.

5. sl n. Average time 9.1 milliseconds. n is a four digit number with the first three digits always zero. Shift the number in the A register n

columns to the left. The first n digits after the sign are lost and n zeros are added at the right.

6. sr n. Average time 8.8 milliseconds. n same as in the sl operation. Shift the number in the A register n columns to the right. The n right hand digits are lost and n zeros are inserted after the sign.

7. cp n. Average time 8.8 milliseconds. n is any four digit number. If an overflow occurred during the preceding operation, replace the number in the order counter by n. Otherwise ignore this order.

8. sp n. Average time 8.8 milliseconds. n is any four digit number. Replace the number in the order counter by n.

9. ca M. Average time 17.5 milliseconds. Clear the address part of the order in the memory box M and add the remainder of the order to the number in the A register.

10. po M. Average time 1117.2 milliseconds. Print out the number in memory box M.

In general some of the above orders would also involve an additional register, the R register. However, in all the computations discussed in this paper the R register will hold nothing but zeros so may be ignored.

2. Problem Solution. In the following discussion marks used to distinguish a thing from its name will be omitted unless necessary to avoid ambiguities.

The operations on or between sentences to be considered are: negation, conjunction, disjunction, material implication, and equivalence. For purposes of the following discussion the word, combination, will represent any one of the latter four.

It will be convenient to make use of the following conventional terms in describing the problem solution. A schema is the result of writing in a row the following symbols in any order: (,), p_j, and the signs for negation and the various combinations. The sign, p_j, represents the various component sentences, j taking on various integral values to distinguish between them. Only well-formed schemas, i.e. schemas in which the parentheses are used to give meaningful groupings, will be used.

The various p_j's are, by definition, schemas of zero order. A schema of order n + 1 is constructed either by putting a negation sign before a schema of order n (which itself must be set off by parentheses) or by forming a combination of two schemas (each set off by parentheses) one of which is of order n and the other of which is of order not greater than n.

It is assumed that the truth values of all schemas of zero order are available. Computation proceeds first by computing the truth values of schemas of order one. Then from the values of schemas of order zero and of order one the truth values of all schemas of order two can be calculated. This process continues until the complete schema has been evaluated. The result is transferred to the memory, the p_j's are altered to the next set of values to be used, and the process is repeated.

This may be illustrated by the following example. Suppose the fifth order schema

$$((p_1 \rightarrow p_0) \wedge (((p_0 \wedge p_1) \wedge \sim p_3) \rightarrow p_2)) \wedge ((p_2 \rightarrow p_1) \wedge (\sim p_0 \rightarrow \sim p_3))$$

is to be evaluated. (Parentheses around individual variables and their negations are omitted.) Let a_j represent schemas of order one, b_j schemas

of order two, and so on. Then

$$a_1 \leftrightarrow (p_1 \rightarrow p_0) \qquad b_1 \leftrightarrow (a_5 \wedge a_2)$$
$$a_2 \leftrightarrow (\sim p_2) \qquad b_2 \leftrightarrow (a_4 \rightarrow a_2)$$
$$a_3 \leftrightarrow (p_2 \rightarrow p_1) \qquad c_1 \leftrightarrow (b_2 \wedge a_3)$$
$$a_4 \leftrightarrow (\sim p_0) \qquad c_2 \leftrightarrow (b_1 \rightarrow p_2)$$
$$a_5 \leftrightarrow (p_0 \wedge p_1) \qquad d_1 \leftrightarrow (c_2 \wedge a_1)$$
$$e_1 \leftrightarrow (d_1 \wedge c_1)$$

Within any order-group any order of calculation may be used. The order shown is convenient since a_5, b_2, c_2, and d_1 are used only in the succeeding computation and thus need not be transferred to the memory. This possibility should always be exploited since in complicated problems it can lead to appreciable savings in computation time.

It remains to work out the evaluation of the negation and combinations. Many procedures have been tried, and those tabulated below require less computation time than any others considered. The times given include transfer from and to memory which may sometimes be avoided as discussed above. The operations to be performed follow from left to right in sequence. The symbols p and q are used for the component sentences. The results of the operations for the various cases are tabulated below the operation. The symbol (x) indicates "the address in the memory of the number x." The symbol C(M) indicates "the contents of the memory box M." The symbol 0^x1 indicates "x zero's followed by the digit 1."

Negation (52.5 milliseconds)

p	ad(01)	su(p)	tm M	C(M)
00	01	01	00	01
01	01	00	00	00

Conjunction (76.4 milliseconds)

p	q	ad(p)	mh(q)	sl0002	tm M	C(M)
00	00	00	0000	00	00	00
00	01	00	0000	00	00	00
01	00	01	0000	00	00	00
01	01	01	0001	01	00	01

Disjunction (88.0 milliseconds)

p	q	ad(p)	ad(q)	ad(09)	sr0009	sl0008	tm M	C(M)
00	00	00	00	09	$0^0$0	00	00	00
00	01	00	01	10	$0^0$1	01	00	01
01	00	01	01	10	$0^0$1	01	00	01
01	01	01	02	11	$0^0$1	01	00	01

Material Implication (88.0 milliseconds)

p	q	ad(q)	su(p)	ad(10)	sr0009	sl0008	tm M	C(M)
00	00	00	00	10	$0^0$1	01	00	01
00	01	01	01	11	$0^0$1	01	00	01
01	00	00	−01	09	$0^0$1	00	00	00
01	01	01	00	10	$0^0$1	01	00	01

Equivalence (109.8 milliseconds)

p	q	ad(p)	ad(q)	ad(98)	cp0008	ad(01)	cp0009	sl0002	ad(01)	tm M	C(M)
00	00	00	00	98	98	99	99	00	01	00	01
00	01	00	01	99	99	1.00	00			00	00
01	00	01	01	99	99	1.00	00			00	00
01	01	01	02	1.00	00				01	00	01
Order counter		0001	0002	0003	0004	0005	0006	0007	0008	0009	

It will be noticed that 01 is the truth value of a true sentence and 00 is the truth value of a false sentence. The zero in the first place simplifies the computation of the implication and has no effect on other computations.

3. Handling of Results. Since results consist of a single digit, it is possible to store ten of them in one memory box. Thus the first answer may be put in the left hand digit position, the second in the next to the left and so on until the memory box is filled. Then the contents of the box may be printed out, the box cleared, and the process repeated.

Storing ten answers in each box reduces the number of print operations by a factor of ten which is desirable since printing is by far the slowest of the machine operations.

Printing during computations makes the results available during computation which may be convenient. Also if the machine breaks down during computation, it will not be necessary to recalculate completely.

4. Altering Data. In order to make use of the advantages of automatic machine calculation, the machine itself should compute all but the initial data. In this problem the data are in the form of (binary) truth values and it is desirable to carry out the computation for all possible combinations of truth values.

Let the truth value (0 or 1) of each component sentence be p_j where j ranges over the values from 0 to $n-1$, n being the number of component sentences to be considered. Construct a binary number k the left hand (or most significant) digit being the result of writing the value of p_{n-1}, the next most significant digit being the result of writing the value of p_{n-2}, and so on. The least significant digit will be the result of writing the value of p_0. All combinations are covered if k starts as zero and then is increased by unity for each computation until every digit of k is one.

In machines using binary arithmetic this procedure can be worked out quite readily. The machine being considered here uses decimal arithmetic which complicates the problem somewhat. The procedure finally adopted makes use of the theorem[2] which follows. For economy in writing, "p_j" will be considered equivalent to "the result of writing the value of p_j."

Definition. A binary number k is defined as the array

$$p_{n-1}p_{n-2}\cdots p_1p_0.$$

Theorem. The number k is replaced by $k+1$ if and only if for every j (if $0 \leq j \leq n-1$, then p_j is replaced by $1-p_j$ if and only if for every m (if $0 \leq m < j$ then $p_m = 1$)). Or more compactly

$$k(\mathrm{B})k+1 \leftrightarrow_j (0 \leq j \leq n-1 \rightarrow (p_j(\mathrm{B})1-p_j \leftrightarrow_m (0 \leq m < j \rightarrow p_m = 1)))$$

(It is understood that p_j remains unchanged unless it is replaced by $1-p_j$ by the above rule, and that at every step p_0 is replaced by $1-p_0$.)

This procedure is programmed quite readily.

5. Terminating the Problem. The machine will not stop, of course, unless it has definite instructions to do so. These may be obtained readily by adding another digit p_n to the left of k. This is zero throughout the computation but will change to one when a one is added to the last value of k to be considered. Thus in the preceding section when the largest value of j satisfying the last equivalence equals n, the computation is finished and the machine may be instructed to stop.

The situation may occur in which all possible combinations of truth values need not be considered. For instance, only true or only false values of a particular sentence may be of interest. In this case this value will be held as a constant (and not stored in the memory with the other p's) and n will be one less than the number of component sentences.

6. The Program. The operations discussed in the four preceding sections are embodied in the flow chart, figure 1, and together make up the program. When the flow chart is worked out in sufficient detail, the program is readily

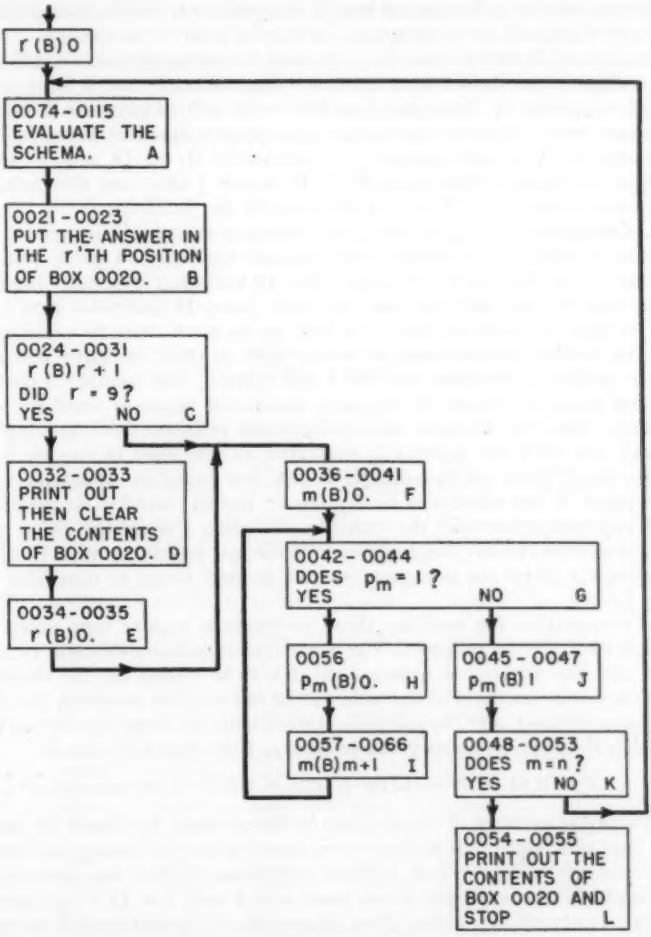

FIGURE 1
FLOW CHART FOR THE PROGRAM

coded or translated into a language to which the machine will respond. A code for this problem is given in Appendix I.

Within the rectangles of the flow chart are first the memory boxes in which the instructions to perform the operation demanded in the rectangle are to be stored (for the particular code given in Appendix I). Secondly the operation to be performed is given. In case the operation to be performed is in the form of a question, the alternative answers to the question indicate the paths by which the computation will proceed. The capital letter in the rectangle identifies the operation for the discussion below.

The expression $x(\mathrm{B})y$ means that at this point y is to be substituted for x wherever x appears in the program. It may be read "x becomes y."

Operations D and E must be performed a number of times equal to the largest digit which is less than 0.1×2^n. Operations H and I must be performed a number of times equal to $2^{n+1} - (n + 2)$. Operation L must be performed once. All other operations must be performed 2^n times.

Operation A is self-explanatory. Operations B, C, D, and E are for handling the results. Operations F, G, H, I, and J take care of altering the data. Operations K and L are for terminating the problem.

7. Conclusion. The program given here was developed and coded with the idea in mind of minimizing the changes necessary in going from one problem to another. It was thought that 19 variables is beyond the limits of practicality (for this machine, at least, since 19 variables would take over six days of machine time). As long as no more than 19 variables are used the instructions in memory boxes 0000 to 0072 inclusive will apply to any problem. Memory box 0073 will contain the number $1.00–0.01n$ (decimal point to the left of the most significant figure of numbers in the memory). This will be used to terminate the computation. Starting with memory box 0074 the schema is evaluated as discussed in section 2. The schema given there as an example is the one coded in Appendix I. The truth value of the schema is finally left in the left hand digit position of the A register. (Note that the routines of section 2 leave the value in the position next to the left hand one so that the last operation must be altered accordingly.) Then the command sp0021 is used to go to operation B of figure 1.

In preparation for working these problems a master tape going only through 0072 will be prepared. For a particular problem this will be duplicated and the additional orders needed will be added on the duplicate. Then the entire contents of the tape will be fed into the machine, the A and R registers cleared, and the machine started with the order counter at 0074.

Using the average times given in section I the problem time is

$$T = 0.91 \times 2^n - 0.18n + 0.75 + 0.077c \times 2^n \text{ seconds},$$

where c is the number of connectives in the schema. It should be emphasized that this is a rough estimate and depends on the assumption that on the average it will take half a drum revolution to find the memory box requested. In the example given here $n = 4$ and $c = 11$ ($\sim p_3$ need be calculated only once) yielding $T = 26$ seconds. (It is understood, of course, that this short example would be worked more economically manually.)

Available data on the Kalin-Burkhart machine are rather sparse, but it would be interesting to compare its operation with that of the program

developed here. It would appear that the Kalin-Burkhart machine will compute the problem a little faster than will the Caldic. The Caldic, however, is rather slow as electronic computers go. Problem preparation time should be about the same. On the other hand the Kalin-Burkhart machine is limited to problems with 12 component sentences and 11 connectives. With the Caldic the number of component sentences is limited only by the time for which the machine is available for the particular problem and there appears to be no practical limitation on the number of connectives.

Appendix I. The Coded Program. These quantities will be fed in sequence into the memory the first into box 0000, the second into box 0001, and so on until the list is exhausted. It is understood that the machine will treat letters as digits and that each number will be filled out with zeros to make ten digits. Every quantity has a plus sign attached. Under each memory box number is given its contents.

0000	0001	0002	0003	0004	0005	0006	0007
0	0	0	0	0	0	0	0
0008	0009	0010	0011	0012	0013	0014	0015
0	0	0	0	0	0	0	0
0016	0017	0018	0019	0020	0021	0022	0023
0	0	0	0	0	sr0000	ad0020	tm0020
0024	0025	0026	0027	0028	0029	0030	0031
ad0021	sl0005	ad0069	cp0032	sr0005	ca0021	tm0021	sp0036
0032	0033	0034	0035	0036	0037	0038	0039
po0020	tm0020	ca0021	tm0021	ca0043	tm0042	ca0047	tm0047
0040	0041	0042	0043	0044	0045	0046	0047
ca0056	tm0056	ad0000	ad0071	cp0056	sl0002	ad0067	tm0000
0048	0049	0050	0051	0052	0053	0054	0055
ad0042	sl0004	ad0073	cp0054	sl0002	sp0074	po0020	STOP
0056	0057	0058	0059	0060	0061	0062	0063
tm0000	ad0042	ad0072	tm0042	ad0047	ad0072	tm0047	ad0056
0064	0065	0066	0067	0068	0069	0070	0071
ad0072	tm0056	sp0042	01	09	10	98	99
0072	0073	0074	0075	0076	0077	0078	0079
000001	96	ad0000	su0001	ad0069	sr0009	sl0008	tm0116
0080	0081	0082	0083	0084	0085	0086	0087
ad0067	su0003	tm0117	ad0001	su0002	ad0069	sr0009	sl0008
0088	0089	0090	0091	0092	0093	0094	0095
tm0118	ad0067	su0000	tm0119	ad0000	mh0001	sl0002	mh0117
0096	0097	0098	0099	0100	0101	0102	0103
sl0002	tm0120	ad0117	su0119	ad0069	sr0009	sl0008	mh0118
0104	0105	0106	0107	0108	0109	0110	0111
sl0002	tm0121	ad0002	su0120	ad0069	sr0009	sl0008	mh0116
0112	0113	0114	0115				
sl0002	mh0121	sl0003	sp0021				

Appendix II. Glossary.

n the number of component sentences in the complete schema.
p_j the jth component sentence.
a_j, b_j, etc. the jth component schema of order 1, 2, etc.
c the number of connectives in the schema.
T estimated computing time in seconds.
1 (or 01) in the proper context the truth value of a true sentence.
0 (or 00) in the proper context the truth value of a false sentence.
x(B)y y is to replace x wherever x appears.
(h) address in the memory of the number h.
C(M) contents of the memory box M.
$\sim$ It is not the case that . . .
$\wedge$ . . . and . . .
$\vee$ . . . or . . .
$\rightarrow$ If . . . then . . .
$\leftrightarrow$ . . . if and only if . . .
$_x$ For every x . . .

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[1] EDMUND C. BERKELEY, *Giant Brains*, New York, 1949, Ch. 9.
[2] GEORGE W. PATTERSON, *Logical Syntax and Transformation Rules*. Moore School of Electrical Engineering, Research Division Report 50–8, Univ. of Pennsylvania, Philadelphia, 1949.

On the Accuracy of Runge-Kutta's Method

1. Introduction. While the accuracy of the most frequently used methods of integrating differential equations is fairly well known, that of the Runge-Kutta method does not seem to be too well established; except for a formula in Bieberbach's text[1] on differential equations there are no references pertaining to the error inherent in the Runge-Kutta method to be found in the standard textbooks on this subject.

Since this method may be employed quite advantageously in many cases of practical interest it is important to have on hand an estimate of the error. The purpose of the following sections is to provide such an estimate. As a comparison shows, the bound derived for this error seems to be somewhat better than the one cited by Bieberbach.

2. Runge-Kutta's Fourth Order Method. In trying to find that solution of the differential equation

$$dy/dx = f(x, y), \qquad y(x_0) = y_0, \tag{1}$$

at $x_1 = x_0 + h$, which agrees with the exact Taylor expansion about x_0:

$$y(x_1) = y_0 + hy_0' + h^2(y_0''/2) + h^3(y_0'''/6) + h^4(y_0^{iv}/24) + h^5(y_0^{v}/120) + \cdots \tag{2}$$

up to the term in h^4, Runge and Kutta developed the following formulae:

$$
\begin{aligned}
(3)\qquad & y(x_1) \approx y_0 + (k_1 + 2k_2 + 2k_3 + k_4)/6, \\
& k_1 = hf_0, \qquad f_0 \equiv f(x_0, y_0), \\
& k_2 = hf(x_0 + h/2, y_0 + k_1/2), \\
& k_3 = hf(x_0 + h/2, y_0 + k_2/2), \\
& k_4 = hf(x_0 + h, y_0 + k_3).
\end{aligned}
$$

To get an estimate of the truncation error inherent in this procedure, one may apply the method first to an interval of length $h_1 = h$, and then integrate over two consecutive intervals of length $h_2 = h/2$. Having the results Y_1, Y_2 of these integrations it is easy to obtain an estimate of the error of the second integration: Since the values Y_1, Y_2 differ from the exact value y_1 by certain errors E_1, E_2:

$$Y_1 = y_1 + E_1, \qquad Y_2 = y_1 + E_2,$$

where

$$E_1 = Ch_1^5, \qquad E_2 \approx 2Ch_2^5 = E_1/16,$$

obviously

$$(4)\qquad E_2 \approx (Y_1 - Y_2)/15.$$

3. Calculation of the Error Term. A more accurate estimate of the error is obtained by a comparison of the exact coefficient $(y_0^v/120)$ of h^5, as it occurs in the Taylor expansion (2), with the approximate one resulting from Runge-Kutta's algorithm (3). Suppose equations (3) have been expressed in the form

$$y(x_1) = y_0 + C_1h + C_2h^2 + C_3h^3 + C_4h^4 + C_5h^5 + \cdots$$

Then $C_i = y_0^{(i)}/i!$, $i = 1, 2, 3, 4$,

$$(5)\qquad C_5 = (y_0^v/5!) + \epsilon,$$

and the following considerations are concerned with the determination of ϵ. In computing the successive total derivatives of a function $u = u(x, y(x))$, $y' = f$, it is helpful to make use of the operator[2]

$$D \equiv \partial/\partial x + f\partial/\partial y.$$

This operator has the following properties:

$$
\begin{aligned}
& D^n = \sum_{k=0}^{n} \binom{n}{k} f^k \partial^n/\partial x^{n-k}\partial y^k, \\
(6)\qquad & D(D^n u) = D^{n+1}u + n(Df)D^{n-1}u_y, \\
& D^0 u = u.
\end{aligned}
$$

Let us now apply the operator D to $f(x, y(x))$ and use the abbreviating notations

$$T^n \equiv D^n f, \qquad S^n \equiv D^n f_y.$$

Then (6) may be expressed as

$$
\begin{aligned}
(7)\qquad & D(T^n) = T^{n+1} + nTS^{n-1}. \\
& DS = S^2 + Tf_{yy}.
\end{aligned}
$$

Applying (7) repeatedly we get

$$\begin{aligned} y' &= f \\ y'' &= Df = T \\ y''' &= D(T) = T^2 + Tf_y \\ y^{iv} &= D(T^2 + Tf_y) = D(T^2) + D(T)f_y + TS \\ &= T^3 + f_yT^2 + 3ST + Tf_y^2. \end{aligned}$$

Similarly,

$$y^v = T^4 + f_yT^3 + 6TS^2 + 4ST^2 + f_y^2T^2 + 3f_{yy}(P)^2 + 7f_yST + f_y^3T,$$

with $(P)^2 = (Df)^2$.

The expansions of the Runge-Kutta's expressions (3) are somewhat more laborious to calculate. First we find that

$$\begin{aligned} k_2 &= hf(x_0 + h/2, y_0 + k_1/2) = h[f + (h/2)f_x + (k_1/2)f_y \\ &\quad + \tfrac{1}{2}((h^2/4)f_{xx} + 2(h/2)(k_1/2)f_{xy} + (k_1^2/4)f_{yy}) + \cdots] \\ &= k_1 + (T/2)h^2 + (T^2/8)h^3 + (T^3/48)h^4 + (T^4/384)h^5 + O(h^6). \end{aligned} \tag{8}$$

Here, as well as in the following, the arguments of f and $f_{x^iy^j}$ are x_0, y_0.

Next we compute

$$\begin{aligned} k_3 &= hf(x_0 + h/2, y_0 + k_2/h) = h[f + (h/2)f_x + (k_2/2)f_y \\ &\quad + \tfrac{1}{2}((h^2/4)f_{xx} + 2(h/2)(k_2/2)f_{xy} + (k_2^2/4)f_{yy}) + \cdots] \\ &= h[f + (1/2)(hf_x + k_2f_y) + (1/8)(h^2f_{xx} + 2hk_2f_{xy} + k_2^2f_{yy}) + \cdots]. \end{aligned}$$

It is convenient to introduce here the operators:

$$\begin{aligned} G_i &\equiv h\partial/\partial x + k_i\partial/\partial y, \qquad i = 1, 2, 3, \\ U_i^n &\equiv G_i^n f = (h\partial/\partial x + k_i\partial/\partial y)^n f. \end{aligned}$$

In terms of the U_i^n we may write

$$k_3 = h[f + (U_2/2) + (U_2^2/8) + (U_2^3/48) + (U_2^4/384) + \cdots]. \tag{9}$$

In order to express k_3 in powers of h we notice from (8) that

$$k_2^2 = k_1^2 + fTh^3 + (1/4)(P + fT^2)h^4 + O(h^5).$$

Further,

$$\begin{aligned} k_2^3 &= k_1^3 + (3/2)f^2Th^4 + O(h^5), \\ k_2^4 &= k_1^4 + O(h^5). \end{aligned}$$

Consequently,

$$\begin{aligned} U_1 &= hT \\ U_2 &= hf_x + k_2f_y \\ &= U_1 + (1/2)f_yTh^2 + (1/8)f_yT^2h^3 + (1/48)f_yT^3h^4 + O(h^5) \\ U_2^2 &= U_1^2 + TSh^3 + (1/4)(ST^2 + Pf_{yy})h^4 + O(h^5) \\ U_2^3 &= U_1^3 + (3/2)TS^2h^4 + O(h^5) \\ U_2^4 &= U_1^4 + O(h^5). \end{aligned}$$

Therefore, by (9),

$$k_3 = k_1 + \tfrac{1}{2}Th^2 + k_{33}h^3 + k_{34}h^4 + k_{35}h^5 + O(h^6), \tag{10}$$

with

$$k_{33} = (2Tf_y + T^2)/8$$
$$k_{34} = (3f_yT^2 + 6ST + T^3)/48$$
$$k_{35} = (4f_yT^3 + 12(ST^2 + Pf_{yy} + TS^2) + T^4)/384.$$

Finally we compute

$$\begin{aligned} k_4 &= hf(x_0 + h, y_0 + k_3) \\ &= h[f + hf_x + k_3f_y + \tfrac{1}{2}(h^2f_{xx} + 2hk_3f_{xy} + k_3^2f_{yy}) + \cdots] \\ &= h[f + U_3 + (U_3^2/2) + (U_3^3/6) + (U_3^4/24) + \cdots]. \end{aligned} \tag{11}$$

Since, by (10),

$$\begin{aligned} k_3^2 &= (k_1 + (T/2)h^2 + k_{33}h^3 + \cdots)^2 \\ &= k_1^2 + fTh^3 + (1/4)(P + fk_{33})h^4 + O(h^5), \\ k_3^3 &= k_2^3, \qquad k_3^4 = k_2^4, \end{aligned}$$

there result the following expressions:

$$\begin{aligned} U_3 &= hf_x + k_3f_y \\ &= U_2 + (1/4)Tf_y^2h^3 + (1/16)f_y(f_yT^2 + 2ST)h^4 \\ U_3^2 &= U_2^2 + (1/2)STf_yh^4 \\ U_3^3 &= U_2^3, \qquad U_3^4 = U_2^4. \end{aligned}$$

Thus it follows from (11) that

$$k_4 = k_1 + Th^2 + k_{43}h^3 + k_{44}h^4 + k_{45}h^5 + O(h^6), \tag{12}$$

with

$$k_{43} = (1/2)(Tf_y + T^2)$$
$$k_{44} = (1/6)(6k_{33}f_y + 3ST + T^3)$$
$$k_{45} = (1/24)(24(k_{34}f_y + Sk_{33}) + 3Pf_{yy} + 6TS^2 + T^4).$$

In the Runge-Kutta expression (3) the coefficient C_5 of h^5 is thus found to be

$$\begin{aligned} C_5 &= (k_{15} + 2k_{25} + 2k_{35} + k_{45})/6 \\ &= (36Pf_{yy} + 60TS^2 + 72f_yST + 36ST^2 \\ &\qquad + 12f_y^2T^2 + 8f_yT^3 + 10T^4)/1152. \end{aligned}$$

It follows that

$$\begin{aligned} \epsilon &= C_5 - (y_0^v/120) \\ &= (-12Tf_y^3 + 9Pf_{yy} + 3TS^2 + 6STf_y - 3ST^2 \\ &\qquad + 3f_y^2T^2 - 2f_yT^3 + (T^4/2))/1440, \end{aligned}$$

or, more explicitly,

$$\begin{aligned} \epsilon = [&-12Tf_y^3 + 3(f_y^2 - S)f_{xx} + 3(2Tf_y + 2ff_y^2 - 2fS - f^2f_{yy})f_{xy} \\ &+ 3(3P + 2Tff_y + f^2f_y^2 - f^2f_{yy})f_{yy} - 2f_yf_{xxx} + 3(f_x - ff_y)f_{xxy} \\ &+ 6ff_xf_{xyy} + (T + 2f_x)f^2f_{yyy} + (T^4/2)]/1440. \end{aligned} \tag{13}$$

4. Estimate of the Error. Let it be assumed, now, that in a certain

region $B(x, y)$ containing (x_0, y_0)

$$\begin{aligned} &|f(x, y)| \leq M, \\ &|f_{x^i y^j}| \leq L^{i+j}/M^{i-1}, \end{aligned} \tag{14}$$

where M, L are positive constants independent of x, y. In that case clearly

$$|T^n| \leq \sum_{j=0}^{n} \binom{n}{j} M^j (L^n/M^{j-1}) = M(2L)^n$$

$$|S^n| \leq \sum_{j=0}^{n} \binom{n}{j} M^j (L^{n+1}/M^j) = L(2L)^n, \quad |P| \leq (2ML)^2.$$

By (13), then

$$|E| \equiv |\epsilon h^5| \leq (73/720) ML^4 h^5. \tag{15}$$

To calculate a suitable value of L one may first compute bounds $L_{i,k}$ for $|f_{x^i y^k}|$, then define quantities

$$L_{i+j} = \max [(L_{i+j,0}/M)^{1/(i+j)}, (L_{i+j-1,1})^{1/(i+j)}, (ML_{i+j-2,2})^{1/(i+j)}, \cdots, (M^{i+j-1} L_{0,i+j})^{1/(i+j)}],$$

for $i + j \leq 4$. Then one may put

$$L = \max (L_1, L_2, L_3, L_4). \tag{16}$$

5. Example. Let us consider the differential equation

$$y' = x + y, \qquad y(0) = 0,$$

and restrict x, y to $0 \leq x \leq 0.2$, $0 \leq y \leq 0.2$. The exact solution is $y = e^x - x - 1$, so that $y(0.2) = 0.02140276$. For $h_1 = 0.2$ Runge-Kutta's method gives $Y_1 = 0.0214$, for $h_2 = 0.1$ it gives $Y_2 = 0.02140257$. Now for the above example we may take $M = 1$, $L = 1$, so that for $h = 0.2$, by formula (15),

$$|E| \approx (1/10)(2/10)^5 = 0.000032;$$

actually the error is less than 0.000003. According to the expression exhibited by BIEBERBACH[1]

$$|E| < 6MNh^5 |N^5 - 1|/|N - 1|, \tag{17}$$

where $|f| < M$, $|f_{x^i y^j}| < N/M^{i-1}$, and, further, $hN < 1$, $aM < b$, where $|x - x_0| < a$, $|y - y_0| < b$. With $N = 1$ formula (17) leads to

$$|E| < 0.0096,$$

which is considerably larger than the error estimate obtained by (15).

6. Systems of Equations. For a system of differential equations

$$y_i' = f_i(x, y_1, y_2), \qquad y_i(x_0) = y_{i0}, \qquad i = 1, 2,$$

the Runge-Kutta formulae become:

$$\begin{aligned} k_{i1} &= hf_{i0}, \quad f_{i0} \equiv f_i(x_0, y_{10}, y_{20}) \\ k_{i2} &= hf_i(x_0 + h/2, y_{10} + k_{11}/2, y_{20} + k_{21}/2) \\ k_{i3} &= hf_i(x_0 + h/2, y_{10} + k_{12}/2, y_{20} + k_{22}/2) \\ k_{i4} &= hf_i(x_0 + h, y_{10} + k_{13}, y_{20} + k_{23}) \\ y_i(x_1) &\equiv y_{i0} + (1/6)(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}). \end{aligned} \tag{18}$$

Proceeding in the manner described in the foregoing sections for the case of a single differential equation, and making use of the abbreviations

$$R^n \equiv D^n f_x, \quad \tau^n = D^n g, \quad \sigma^n \equiv D^n g_y, \quad \rho^n \equiv D^n g_x, \quad \pi \equiv [\tau]^2$$

one obtains for $\epsilon_1 = C_{15} - y^v(x_0, y_{10}, y_{20})/120$ the expression

$$\begin{aligned} 1440\epsilon_1 = (T^4/2) &- 2(f_y T^3 + f_x\tau^3) + 3[T^2(f_y^2 + f_x g_y) + \tau^2(f_y f_x + f_x g_x)] \\ &- 3(ST^2 + R\tau^2) + 12[S(Tf_y + \tau f_x) + R(Tg_y + \tau g_x)] \\ &- 6[T(Sf_y + \sigma f_x) + \tau(Rf_y + \rho f_x)] + 3(TS^2 + \tau R^2) \\ &+ 9(Pf_{yy} + 2T\tau f_{yx} + \pi f_{xx}) \\ &- 12[T(f_y^3 + f_x g_y(2f_y + g_x)) + \tau f_x(f_y^2 + f_y g_x + f_x g_y)] \\ &- 15f_{yx}(T^2 g + \tau^2 f). \end{aligned}$$

Assumptions similar to (14) now permit one to get a bound for $\epsilon_i h^5$.

If, namely, near (x_0, y_{10}, y_{20}),

$$|f_i| \leq M, \qquad |\partial^{p+q+r} f_i/\partial x^p y_1^q y_2^r| \leq L^{p+q+r}/M^{q+r-1}$$

for $0 \leq p + q + r \leq 4$, it is found that

$$|E_i| \leq (973/720)ML^4h^5. \tag{19}$$

For the differential equation $y'' - y = 1$, $y(0) = 0$, $y'(0) = 1$, which is equivalent to the system $y_1' = y_2$, $y_2' = 1 + y_1$, $y_1(0) = 0$, $y_2(0) = 1$, the solution is $y = e^x - 1$. Thus $y(0.1) = .1051709$. With $h = 0.1$, R.K.'s method gives $y(0.1) = .1051707$, whence $E = 2 \cdot 10^{-7}$. In the region $0 \leq x \leq 0.1$, $0 \leq y_1 \leq 0.11$, $0 \leq y_2 \leq 1.11$; above estimate (19) asserts that $|E_i| \leq 1.5 \cdot 10^{-5}$.

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[1] L. BIEBERBACH, *Theorie der Differentialgleichungen*. New York, 1944, p. 54.
[2] C. RUNGE & H. KÖNIG, *Vorlesungen über numerisches Rechnen*. Berlin, 1924.

RECENT MATHEMATICAL TABLES

878[F].—H. CHATLAND & H. DAVENPORT, "Euclid's algorithm in real quadratic fields," *Canadian Jn. Math.*, v. 2, 1950, p. 289–296.

The tables in this paper are rather special and are used to prove a result of INKERI[1] that the field $k(m^{\frac{1}{2}})$ is not Euclidean for

$$m = 193{,}241{,}313{,}337{,}457{,}601.$$

There are six tables corresponding to these values of m. Each gives a complete period of a periodic algorithm showing certain inequalities which establish the non-existence of Euclid's algorithm in each field considered.

[1] K. INKERI, "Über den Euklidischen Algorithmus in quadratischen Zahlkörpern," Acad. Sci. Fennicae, *Annales*, v. 41, 1947, p. 5–34.

879[F].—A. GLODEN, "Factorisation de nombres $N^{16} + 1$," *Euclides*, v. 10, 1950, p. 157.

A small table gives the complete factorization of $N^{16} + 1$ for $N = 2(1)13$, 21,27,32 with references to authors. Two additional entries, for $N = 16$ and $N = 64$, might have been given:

$$16^{16} + 1 = 274177 \cdot 67280421310721$$
$$64^{16} + 1 = 641 \cdot 6700417 \cdot 18446744069414584321.$$

The first of these is the historic factorization by LANDRY of the 6-th FERMAT number. For the second see Amer. Math. Soc. *Bull.*, v. 36, 1936, p. 849. A third factorization corresponding to $N = 36$

$$36^{16} + 1 = 2753 \cdot 145601 \cdot 19854979505843329$$

has been announced by HOPPENOT and KRAITCHIK [*Sphinx*, v. 4, 1934, p. 47] but no really satisfactory proof of the primality of the largest factor has been given.

D. H. L.

880[F].—HELMUT HASSE, *Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörper*. Akad. Wissensch., Berlin, *mat. nat. Kl., Abhandlungen*, 1948, n. 2, Berlin, 1950. vii + 95 p., 29.5 × 21 cm. 14 marks.

This paper contains tables for the fundamental units, the cyclotomic units and the class numbers of all cyclic cubic and all real cyclic biquadratic fields with conductor $\leqq 100$.

A systematic method is developed for finding these quantities generally in such fields; it is applicable to the case of non-real cyclic biquadratic fields. The work of H. Bergström[1] is used and generalized.

The analytic expression for the class number depends on the index of the system of cyclotomic units with respect to the system of all units in the maximum real subfield. For real fields this is the field itself. A cyclic cubic field is always real and has two fundamental units, a cyclic biquadratic field has three or one according as it is real or imaginary. It is shown that, for cubic fields, there exists a fundamental unit E such that -1 and the conjugates of E generate all the units. For real biquadratic fields the existence of a relative fundamental unit E is shown which has relative norm ± 1 with respect to the quadratic subfield (which is always real). The conjugates of E, the fundamental unit of the quadratic subfield and -1 generate a subgroup of index 1 or 2 in the complete group of units. In both cases E is characterized by a minimum property of its coordinates with respect to the integral base, just as the fundamental unit of a real quadratic field corresponds to the smallest positive solution of the Pellian equation.

The integral base is chosen in a special way by using ideas of class field theory. This is done in order to determine it in an invariant way. This can be regarded as a generalization of the fact that in the quadratic case the discriminant determines the base.

O. TAUSSKY

NBS

[1] H. BERGSTRÖM, "Die Klassenzahlformel für reelle quadratische Zahlkörper mit zusammengesetzter Diskriminante als Produkt verallgemeinerter Gausscher Summen," *Jn. reine angew. Math.*, v. 186, 1944, p. 91–115.

881[F].—E. H. NEVILLE, editor, *The Farey Series of Order 1025, Displaying Solutions of the Diophantine Equation* $bx - ay = 1$. *Designed and compiled by* E. H. N. (*Royal Society Mathematical Tables*, volume 1). Cambridge, University Press; published for the Royal Society, 1950. xxxii, 405 p. 21.5 × 28 cm. There is also a loose card, 20.2 × 27.2 cm., giving the complete factorization, on one side of the numbers 1–549, and on the other of 500–1049. £5 s5. Cambridge University Press, $18.50, New York.

This volume was accepted for publication by the British Association Tables Committee in 1947. In the following year the responsibility which the British Association had exercised since 1871 through that Committee was transferred to the Royal Society which set up a Committee of its own to take over the work [see *MTAC*, v. 3, p. 333–340]. The publication of the volume under the CUNNINGHAM bequest for the production of new tables in the theory of numbers is especially appropriate.

The so-called FAREY series, F_n, of order n, is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n; the numbers 0 and 1 are included in the forms 0/1 and 1/1. Thus, F_7 is

$$\frac{0}{1}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{1}{1}$$

The fraction $\frac{1}{2}$ is the middle term of every Farey Series; if two terms are equidistant from this, their sum is unity; they have the same denominator and this denominator is the sum of the two numerators. To emphasize this property of the series, as well as to economise in presentation, F_7 would be written:

0	1	1	1	1	2	1	2	3	1
1	7	6	5	4	7	3	5	7	2
1	6	5	4	3	5	2	3	4	1

Pages 1–400 of the volume before us are taken up with the presentation of the 319765 terms of F_{1025}, written in this form. Except for the last page there are 400 terms on each page, of 20 lines, each line containing 20 term-pairs, separated as two groups of 10, and the whole arranged in 8 groups of 50 term-pairs. The first three and the last three printed term-pairs of F_{1025} as given are

0	1	1		511	512	1
1	1025	1024	and	1023	1025	2
1	1024	1023		512	513	1

It is clear that all terms of an F_m, $m < n$, are contained in the series of terms of F_n.

If a_1/b_1, a_2/b_2, a_3/b_3 are any three consecutive terms of F_n

I. $(a_1 + a_3)/(b_1 + b_3) = a_2/b_2$.

II. $(a_1/b_1) - (a_2/b_2) = 1/(b_1b_2)$ or $(a_2/b_2) - (a_3/b_3) = 1/(b_2b_3)$, b_1, b_2, b_3 being relatively prime.

III. Any term between a_s/b_s and a_t/b_t is of the form $(pa_s + qa_t)/(pb_s + qb_t)$, where p and q are integers, and is irreducible if p, q are relatively prime.

IV. The total number of terms in F_n is $\Phi(n) + 1$, where $\Phi(n)$ is the sum function of Euler's $\phi(n)$.

The work under review contains three appendices (which I shall call A, B, and C).

Appendix A (p. 402–403) gives the Farey series of order 50 with decimal equivalents to 5D.

Appendix B (p. 404) displays all the terms of F_{64}, in order to give the reader a general idea of a Farey series which may not be deduced from the 400 pages filled with F_{1025}.

Appendix C (p. 405) exhibits the "Farey integer-series" of order 100, that is, the denominators, in order, of all the terms of the series in F_{100}. The page has to be read twice (once forward and once backward), in order to cover all terms.

Let us now pause to consider some historical facts connected with the series under consideration, insofar as they have bearing on the contents of the work under review.

α. In *Ladies' Diary* for 1747, p. 34, Question 281, J. May, Jr., of Amsterdam, offered the problem: "It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the greatest denominator be less than 100." Since reducible fractions are not excluded the number here would be in excess of the number of terms in F_{99}. Solutions are presented in *Ladies' Diary* for 1748, p. 23, and for 1751, p. 30.

β. In *Journal de l'École Polyt.*, cahier 11, v. 4, 1802, p. 364–368, Citizen Haros writes on the topic "Tables pour évaluer une fraction ordinaire avec autant de décimales qu'on voudra; et pour trouver la fraction ordinaire la plus simple, et qui approche sensiblement d'une fraction décimale."

Haros here proved the general results I and II stated above.

γ1. In 1816 Henry Goodwyn (1745–1824) printed privately and distributed some copies of his *The First Centenary of a Series of Concise and Useful Tables of all the complete Decimal Quotients, which can arise from dividing a unit, or any whole Number less than each Divisor by all Integers from 1 to 1024.* This publication ("specimen" he calls it) contained xiv + 18 pages. Page i is simply the title page quoted above; p. ii is blank; on p. iii is an introductory statement signed by Goodwyn and dated March 5th, 1816. The 18 pages of tables simply contain 6D equivalents of the various fractions with denominators successively 1(1)100. From this series of tables one could get no ready idea of the order of successive terms in a so-called Farey series.

γ2. At London in 1818 Goodwyn sent out a publication on the upper part of the title page of which is the italics title quoted above; then follows immediately: *To which is now added a Tabular Series of Complete Decimal Quotients for all the Proper Vulgar Fractions of which when in their lowest terms, neither the Numerator nor the Denominator is greater than 100: with the equivalent vulgar fractions prefixed.* This is repeated as title page (i) of the separately paged addition. viii + 32 p. Here we have, p. 1–14, not only all the terms in the first half of F_{100}, and their 10D equivalent values, but also the means for writing down at once the remaining terms of the series and their decimal equivalents.

Hence, this table offers the means for (a) reading off all the results of Appendix C in the table under review; (b) displaying the Farey series of order 100 rather than of order 64 in Appendix B; (c) reading off the Farey series of order 100 with decimal equivalents rather than the series of order 50 with diminished decimal approximations in Appendix A.

Furthermore, on page v of this 1818 publication of Goodwyn we find in his survey of the series: "every Fraction in the Series (since they are all in their lowest terms), is the *exponent* of the *ratio* of the *sum* of the *numerators* to the sum of the *denominators*, of the Fractions occurring *next before* and *after* it." This is just one of the results found by Haros 16 years before. But in this connection Goodwyn does not mention the name of anyone else.

δ. It will presently appear why we must now turn back in our chronological treatment. In *Phil. Mag.*, v. 47, May 1816, p. 385–386, the geologist John Farey (1766–1826; see *Dict. Nat. Biog.*) contributed a letter to the editor. The title for the heading to this letter is "On a curious property of vulgar fractions." The letter commences: "On examining lately, some very curious and elaborate Tables of 'Complete decimal Quotients' calculated by Henry Goodwyn, Esq. of Blackheath, of which he has printed a copious specimen, for private circulation among curious and practical calculators, preparatory to the printing of the whole of those useful tables, if sufficient encouragement, either public or individual, should appear to warrant such a step: I was fortunate while so doing, to deduce from them the following general property; viz.,"—and then follows the Haros-Goodwyn theorem, with numerical illustrations from F_5 and F_{99}.

What deductions may we now make? Farey was writing about May 1816 and refers to Goodwyn's privately distributed publication of the previous March, and also, to tables of "Complete Decimal Quotients." It is obvious that by 1816 Farey must have seen both tables of the 1818 publication of Goodwyn, to each of which the above quoted title applies, in order to make public his deduction (impossible from the 1816 table alone). We have noted that Goodwyn formulates this same result.[1]

Farey's letter was reported in Soc. Philomathique de Paris, *Bull. d. Sci.*, 1816, p. 112, but Farey's name is not mentioned. This communication attracted the attention of Cauchy, who gave a proof, and extension, in the same volume of the *Bulletin*, p. 133–135. As we have seen these results, I–II, were already published by Haros 14 years earlier.

The result III was proved by Edward Sang (R. Soc. Edinb., *Trans.*, v. 28, 1879, p. 287). The name "Farey Series," which, as we have seen, has not the slightest justification historically, seems to have been first used by Sylvester, who also tabulated IV (*Phil. Mag.*, s. 5, v. 15, 1883, p. 251f.). Well knowing all the facts in question Neville would have served scholarship better by entitling his book: *The Haros Series of Order 1025*, or *The Goodwyn Series of Order 1025*.

The table in clockmaker Achille Brocot's *Calcul des Rouages par Approximation. Nouvelle Méthode*, Paris, 1862, p. 49–89, is a display of F_{100}, with each term approximated to 10D; the first half is identical with Goodwyn's 1818 table. (γ2)

In Hütte, *Hilfstafeln zur I. Verwandlung von echten Brüchen in Dezimalbrüche* . . . Third newly revised edition, Berlin, 1922, p. 24–61, J. T. Peters checked and extended "Brocot's" table from 10D to 11D.

In his *Manual of Gear Design, Section One*, New York, 1935, p. 148–169, EARLE BUCKINGHAM proposed to give a table of F_{120} to 8D, but there are numerous omissions and errors; see *MTAC*, v. 1, 1943, p. 92.

On the other hand, the terms of F_{120} in R. M. PAGE, *14000 Gear Ratios*, New York, 1942, are complete and the corresponding 11D values more reliable. [See *MTAC*, v. 1, 1943, p. 21–23; 1944, p. 326–329, 430.]

Until the publication of Neville's volume the most extensive F_n to be published was that for $n = 120$. The statement by D. H. L. in his *Guide to Tables in the Theory of Numbers*, Washington, 1941, p. 8, that Goodwyn gave F_{1000} is therefore incorrect; see *MTAC*, v. 1, p. 372, 1945.

All solutions (x, y) of the equation

$$bx - ay = c, \qquad (a, b, c, \text{integers}, a, b, \text{coprimes}) \tag{1}$$

are given by the formulas

$$x = ka + Xc, \qquad y = kb + Yc, \qquad k \text{ any integer},$$

if

$$bX - aY = 1. \tag{2}$$

Hence, a/b and X/Y are consecutive terms of a Farey series, and solutions (X, Y) of (2), and hence of (1) are known if the neighbours of a/b in F_b are known. Neville's Introduction explains, with numerical illustrations, not only the processes by which solutions of any equation (1) may be obtained, however much b exceeds n, but also how to find desired terms in F_{1021}. A "Decimal Index" table on p. xxviii–xxix, and the loose card, are useful in this connection.

Solutions of (1) have important applications in different parts of the theory of numbers.

The two earlier tables for finding solutions of (2) are by A. L. CRELLE (in his *Journal*, v. 42, 1851, p. 304–313), and by A. J. CUNNINGHAM (in his *Quadratic and Linear Tables*, London, 1927, p. 134–157). Crelle gives immediate solutions of (2) for $a < b \leq 120$. The results of Cunningham's table, for $b < 100$, $a < 100$, are therefore included in Crelle's. Within its range the Crelle table would naturally furnish results more readily than Neville's tables would.

To ensure accuracy in this splendid new volume of the British Mathematical Tables Committee more than 40 people collaborated with Professor Neville. Hence we must wonder why neither they nor the printer noticed that on page 399 are ten numbers which should have been in ordinary rather than in blackface type.

This volume is dedicated "To the memory of SRINIVASA RAMANUJAN," Neville's friend, since "to every mathematician of our time Farey's series" recalls his name.

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[1] Not only the publications of GOODWYN to which we refer, but others, have been discussed in thorough fashion by J. W. L. GLAISHER: (i) *Report of the Comm. on Mathematical Tables*, 1873, p. 31–33, 150; (ii) "On circulating decimals with special reference to Henry Goodwyn's 'Table of circles' and 'Tabular series of decimal quotients' (London 1818–1823)," Cambridge Phil. Soc., *Proc.*, v. 3, 1878, p. 185–206; (iii) "On a property of vulgar fractions," *Phil. Mag.*, s. 5, v. 7, 1879, p. 321–336. We have already quoted DEMORGAN's

Statement of 1861 [*MTAC*, v. 2, p. 87] concerning Goodwyn: "His manuscripts, an enormous mass of similar calculations, came into the possession of Dr. OLINTHUS GREGORY, and were purchased by the Royal Society at the sale of his [Gregory's] books in 1842." In the above publications of Glaisher it is stated that no trace of the papers could be found at the Royal Society. Neville's statement, p. xi, "Goodwyn left a mass of papers, no one knows what became of them," is therefore slightly misleading.

882[F].—BENGT PERSSON, "On a Diophantine equation in two unknowns," *Arkiv Mat.*, v. 1, 1949, p. 45–57.

The table on p. 57 gives all solutions of the diophantine equation $x^2 + x + (p + 1)/4 = y^3$ for all values of the prime $p < 100$ of the form $4n + 3$, except the value $p = 3$ which is handled separately on p. 45 of the paper.

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883[F].—B. VAN DER POL & P. SPEZIALI, "The primes in $k(\rho)$," K. Nederl. Akad. Wetensch., *Proc.*, v. 54 A, 1951, p. 9–15, 1 plate.

The primes of the quadratic field $k(e^{2\pi i/3})$ are presented in graphical form. Each integer in the field is assigned to the center of a hexagon. The totality of integers thus fills the plane. Those integers which are primes are indicated by black hexagons, the non-primes by white hexagons. The plate inserted between p. 13 and 14 gives a graphical representation of all primes in the field whose norms are less than 10000. The beautiful symmetrical diagram contains 7416 black hexagons out of approximately 36276 hexagonal cells giving a density of primes of about one out of five.

An enlargement of the central part is given in another figure. The paper contains a table of solutions (α, β) of the quadratic partition

$$p = 6m + 1 = \alpha^2 + 3\beta^2 \qquad \text{for} \qquad p < 10000.$$

There are two misprints, communicated by the author:

for	$p = 1559$	*read*	1549
for	$p = 7459$	*read*	7549

This table was used to construct the diagram. The authors were un are that previous tables of this extent exist. JACOBI's table[1] for $p \leqslant 1'$ 17 is more than a century old. Other tables by REUSCHLE[2] and CUNNI AM[3] extend to $p = 13669$ and $p = 125683$ respectively. [See also UM 4.] A comparison of Cunningham's table with the present one r als no discrepancy.

A similar diagram for the primes of the field $k(i)$, based on squares, was published by van der Pol.[4]

D. H. L.

[1] K. G. J. JACOBI, "Über die Kreistheilung und ihre Anwendung auf die Zahlentheorie," *Jn. reine angew. Math.*, v. 30, 1846, p. 166–182.

[2] K. G. REUSCHLE, *Mathematische Abhandlung, enthaltend: Neue Zahlentheoretische Tabellen*, Stuttgart, 1856.

[3] A. J. C. CUNNINGHAM, *Quadratic Partitions*. London, 1904; *Quadratic and Linear Tables*. London, 1927.

[4] B. VAN DER POL, Verslagen van de Maatschappij Diligentia, The Hague, 1946. A copy of this diagram woven in red and white squares hangs on the wall of the reviewer's study, a gift from the author.

884[H].—W. P. THIJSEN, "Over de geïtereerde machtsverheffing," *Simon Stevin*, v. 27, p. 177–192, 1950.

The table on p. 185 gives approximate solutions of the equations:

$$(1) \qquad t = x^t$$

$$(2) \qquad t = x^{x^t}$$

for given x. Equation (1) has two solutions:[1] $t = \lambda, \mu$; $1 < \lambda < e < \mu < \infty$. Similarly equation (2) has two solutions: $t = \sigma, \tau$; $0 < \sigma < e^{-1} < \tau < 1$. 5D values (mostly) are given for λ, and 4D values for σ, τ, all for $x = 0(.01).06$, $e^{-e} = .06599$, $.1(.1)1$, 1.2, $e^{1/e} = 1.447$. The values of μ corresponding to $x = 1, 1.2, e^{1/e}$ are ∞, 14.77, e respectively.

On p. 184 there are graphs[1] illustrating $\lambda, \mu, \sigma, \tau$ as functions of x.

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[1] The curve $t = x^t$ has been frequently studied before. For example: (a) J. F. C. HESSEL, "Über das merkwürdige Beispiel einer zum Theil punctirt gebildeten Curve das der Gleichung entspricht $y = \sqrt[x]{x}$," *Archiv Math. Phys.*, v. 14, 1850, p. 169–187; H. SCHEFFER, "Über die durch die Gleichung $y = \sqrt[x]{x}$ dargestellten Curven," *Archiv Math. Phys.*, v. 16, 1851, p. 133–137.

The role which $\xi = \omega^\xi$ plays in CANTOR's theory of transfinite numbers will be recalled; see, for example, G. Cantor, *Math. Annalen*, v. 49, 1897, p. 242–246 (also the English translation of P. E. B. JOURDAIN, Open Court, 1915, p. 195–201).

885[I].—H. E. SALZER, "Coefficients for polar complex interpolation," *Jn. Math. Phys.*, v. 29, 1950, p. 96–104.

This article is supplementary to two earlier papers,[1] both of which are devoted to the computation of the coefficients of interpolation formulas for the representation of a function $f(z)$ defined at n successive points z_k, where k ranges from $[\frac{1}{2}n - \frac{1}{2}]$ to $[\frac{1}{2}n]$, the points being spaced equally along the arc of a circle about the origin of coordinates.

In the first of these papers the interpolation formula was written in the form

$$f(z) \sim \sum L_k^{(n)}(P) f(z_k),$$

where the point P was defined to be $P = (z - z_0)/(z_1 - z_0) = p + iq$, and where P^n was represented by $p_n + iq_n$. Explicit formulas were given for the real and imaginary parts of the coefficients $L_k^{(n)}$, which are functions not only of p_n and q_n, but also of θ, since $z = \rho e^{i\theta}$. The three cases of 3-point, 4-point, and 5-point interpolation were treated.

In the second paper the interpolation formula appeared as follows:

$$f(z) \sim (\sum a_k f_k)/\sum a_k,$$

where we write

$$a_k = [(z_k - z_0) \cdots (z_k - z_{k-1})(z_k - z_{k+1}) \cdots (z_k - z_{n-1})(z - z_k)]^{-1}.$$

If new quantities $A_k^{(n)} = (z - z_k)a_k$ are introduced, and if $z_0 = 0$, $z_1 = 1$, $z_2 = 1 + e^{i\theta}$, $z_3 = 1 + e^{i\theta} + e^{2i\theta}$, $z_{-1} = -e^{-i\theta} - e^{-2i\theta}$, etc., then $A_k^{(n)}$ can be computed as functions of θ. This article gives the explicit values for these functions for 3-point to 9-point interpolation inclusive.

The article under review provides tables of the functions $L_k^{(n)}$ and $A_k^{(n)}$, computed in the first case to 8D and in the second case to 9D, with some doubt as to the ninth place, over the range $\theta = 1°, 5°, 10°, 15°, 20°$, and $30°$.

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[1] H. E. SALZER, "Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs," *Jn. Math. Physics*, v. 24, 1945, p. 141–143, *MTAC*, v. 2, p. 73, and "Alternative formulas for direct interpolation of a complex function tabulated along equidistant circular arcs," *Jn. Math. Physics*, v. 26, 1947, p. 56–61.

886[K].—R. L. ANDERSON & T. W. ANDERSON, "Distribution of the circular serial correlation coefficient for residuals from a fitted Fourier series," *Annals Math. Stat.*, v. 21, 1950, p. 59–81.

The model for observations x_i, $i = 1, 2, \cdots, N$, here used is

$$x_i - \mu_i = \rho(x_{i-1} - \mu_{i-1}) - u_i,$$

with $x_0 \equiv x_n$ and $\mu_0 \equiv \mu_n$, in which the $\{u_i\}$ are random disturbances each independently and normally distributed with zero mean and variance σ^2, and μ_i is a linear form in Fourier terms. The coefficients of the Fourier terms are the usual regressions of the x_i on these terms. The fitted values of the μ_i are denoted by m_i and the serial correlation coefficient

$$R = \sum_{i=1}^{N} (x_i - m_i)(x_{i-1} - m_{i-1}) \Big/ \sum_{i=1}^{N} (x_i - m_i)^2$$

(with lag 1), where $m_0 \equiv m_n$, is used to test hypotheses concerning ρ. Under the null hypothesis of zero serial correlation ($\rho = 0$), Table I, p. 65, gives to 3D the exact 5% and 1% points for R for P periods used in the fitted series for the following cases: $P = 2$, $N = 6(2)60$; $P = 2, 4$, $N = 8(4)100$; $P = 2, 3, 6$, $N = 12(6)150$; $P = 2, 12/5, 3, 4, 6, 12$, $N = 24(12)300$. Table II, p. 66–67, gives exact 5% and 1% significance points for R (the distribution of R is asymmetric in these cases) for single periods for $P = 3$, $N = 6(6)150$; $P = 6$, $N = 12(6)150$; $P = 4$, $N = 8(4)100$, $108(12)144$; $P = 12$, $N = 12(12)300$.

C. C. C.

887[K].—D. G. CHAPMAN, "Some two sample tests," *Annals Math. Stat.*, v. 21, 1950, p. 601–606.

Let s represent the difference of two independent variables, each of which has a t-distribution with $n - 1$ degrees of freedom. Table I contains values of $P(0 \leqq s \leqq s_0)$; the values of n and s_0 considered are: $n = 2(2)12, \infty$; $s_0 = .5(.5)8(2)12, 21, 30, 50, 100$. Table II contains values of s_0 such that $P(|s| \geqq s_0) = .01, .05$ for $n = 2(2)12, \infty$.

Another place where 1 percent and 5 percent points for the absolute difference of two independent t-statistics can be obtained is in the 1948 edition of FISHER & YATES' tables.[1] Let t_1 have a t-distribution with n_1 degrees of freedom while t_2 is independent of t_1 and has a t-distribution with n_2 degrees of freedom. Let $d = (t_1 - t_2)\cos\theta$. Table V_1 (by P. V.

SUKHATME) contains values of d_0 such that $P(|d| \geq d_0) = .01, .05$ for $n_1, n_2 = 6, 8, 12, 24$, and $\theta = 0°(15°)90°$.

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[1] R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural, and Medical Research*. London, 1948.

888[K].—JEROME CORNFIELD & NATHAN MANTEL, "Some new aspects of the application of maximum likelihood to the calculation of the dosage response curve," Amer. Stat. Assn., *Jn.*, v. 45, 1950, p. 181–210.

Table 1 (p. 185–188) gives Z/P, $-Z/Q$, $Z^2/Q^2 - ZY/Q$, and $ZY/P + Z^2/P^2$, all to 5D, for $X = 5(.01)10$, and for $X = 0(.01)5$ by symmetry. $X - 5 = Y$ is a normal deviate, Z being the corresponding ordinate; P is the probability of a deviate less than Y, and Q that of a larger deviate. The purpose of the table is to minimize the arithmetic of the maximum likelihood estimate of the probit regression. There are detailed instructions for calculation.

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889[I, K].—D. B. DELURY, *Values and Integrals of the Orthogonal Polynomials up to n = 26*. Toronto, Published for Ontario Research Foundation by University of Toronto Press. 1950, v + 33 p. 17.5 × 25.2 cm. $1.25.

The polynomials tabulated are those of CHEBYSHEV[1] and GRAM and are the FISHER & YATES[2] polynomials denoted by $\xi_r'(x) = \xi_r'(x, n)$.

The values of $\xi_r'(x, n)$ are tabulated for $r = 0(1)n - 1$ and for $n = 3(1)26$.

The sums of squares of the ξ's are given at the foot of each column. For $r \leq 14$ and $n \leq 26$ values of

$$I_r = \int_0^{n-1} \xi_r'(x, n)\, dx$$

and

$$J_r = \int_{-\frac{1}{2}}^{n-\frac{1}{2}} \xi_r'(x, n)\, dx$$

are tabulated.

The tables presented in the book were prepared for the purpose of furnishing an arithmetical basis for a numerical integration procedure appropriate to situations in which the observed or measured ordinates are subject to a random error.

Of considerable interest is the discussion concerning the fitting of a function in two variables by means of orthogonal polynomials.

Two examples of the use of the tables are given. The first deals with evaluating an integral over a range of equally spaced data. The second example shows how the total number of trees on a given area can be estimated from counts on square plots situated centrally on square blocks within the area. The orthogonal polynomials are used to determine the

equation of the surface. The equation is then integrated to yield the desired estimate.

It is of interest to point out that a table of orthogonal polynomials and their derivatives has been prepared by W. F. BROWN, JR. & C. W. DEMPSEY [*MTAC*, v. 4, p. 224–5].

A comparison between the values of the polynomials tabulated by DeLury and by Brown & Dempsey show no discrepancies. It is to be regretted that the values of the ratio of ξ' to ξ are not given in the book.

The book is clearly printed. It should be a welcome addition to the library of statisticians and mathematicians.

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Further values of these orthogonal polynomials have been given by ANDERSON & HOUSEMAN[3] for $n = 3(1)104$, $r = 1(1)\min(5, n)$ and by VAN DER REYDEN[4] for $n = 5(1)13$, $r = 1(1)$ various values less than 9 and for $n = 14(1)52$, $r = 1(1)9$.

The introduction describes the uses of the table in fitting polynomials by least squares, in finding integrals of the resulting polynomials, in carrying out the associated analyses of variances and in determining confidence intervals in one and in two independent variables.

Values of ξ_r' for $r \leq 5$ are said to be reprinted from FISHER & YATES.[2] The author states that "The director of the Onderstepoort Veterinary Research Institute and Dr. D. VAN DER REYDEN[4] also have kindly granted permission to reproduce portions of the tables which appeared in the *Onderstepoort Journal*." The present tables are, however, correct in two places where minus signs are missing from van der Reyden's table (for $n = 17$, $r = 5$, $\xi_1' = -3$ and $n = 18$, $r = 5$, $\xi_1' = -1$. Van der Reyden gives values of the polynomials for $x \leq \bar{x}$). And a common factor in the values for $n = 25$, $r = 8$ in van der Reyden's table has been removed.

For more complete discussions of these orthogonal polynomials see the review by EISENHART [*MTAC*, v. 1, p. 148–150] and papers by BIRGE,[5] WEINBERG[6] and van der Reyden[4] and the references given in these places. For punched card procedures using such tables as the one under review see a paper by LILA KNUDSEN.[7]

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[1] P. L. CHEBYSHEV, "Sur l'interpolation par la méthode des moindres carrés," Akad. Nauk, Leningrad, *Mémoires*, s. 7, v. 1, no. 15, 1859, p. 1–24. *Oeuvres*, v. 1, p. 471–498.

[2] R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*. Third ed., London, 1948. (First ed., 1938.)

[3] R. L. ANDERSON & E. E. HOUSEMAN, *Tables of Orthogonal Polynomial Values Extended to N = 104*. Ames, Iowa, Iowa State College of Agriculture and Mechanic Arts, 1942. (*Research Bulletin* 297, p. 593–672, Agricultural Experiment Station, Statistical Sections.) [*MTAC*, v. 1, p. 148–150, v. 5, p. 81.]

[4] D. VAN DER REYDEN, "Curve fitting by the orthogonal polynomials of least squares," *Onderstepoort Journal of Veterinary Science and Animal Industry*, v. 28, 1943, p. 355–404.

[5] RAYMOND T. BIRGE, "Least-squares' fitting of data by means of polynomials," *Review of Modern Physics*, v. 19, 1947, p. 298–347.

[6] J. W. WEINBERG, "Mathematical appendix," *ibid.*, p. 348–360.

[7] LILA F. KNUDSEN, "A punched card technique to obtain coefficients of orthogonal polynomials," Am. Stat. Assn., *Jn.*, v. 37, 1942, p. 496–506.

890[K].—A. K. GAYEN, "The distribution of the variance ratio in random samples of any size drawn from non-normal universes," *Biometrika*, v. 37, 1950, p. 236–255.

The author is interested in the non-normal sampling distribution of the two statistics: w = the ratio of two estimates of the variance obtained in a one-way classification for the analysis of variance (homogeneity), v = the variance ratio of two independent samples (compatibility). To introduce non-normal populations the EDGEWORTH series,

$$f(x) = \varphi(x) - \frac{\lambda_3}{6}\varphi^{(3)}(x) + \frac{\lambda_4}{24}\varphi^{(4)}(x) + \frac{\lambda_3^2}{72}\varphi^{(6)}(x),$$

is used. With an Edgeworth series population the author evolves the frequency densities of the two statistics. These distributions are then related to their corresponding "normal" sampling distributions and correction terms for the skewness and excess are determined. The coefficients of λ_3 $(= \sqrt{\beta_1})$ and λ_4 $(= \beta_2 - 3)$ are tabulated in Table 4 (p. 252–255) to 4D for the 5 percent points of Fisher's Z for the degrees of freedom $\nu_1 = 1(1)6, 8, 12, 24, \infty$; $\nu_2 = 1(1)6, 8, 12, 20, 24, 30, 40, 60, 120, \infty$. These tables enable one to compute the error one would make in using the ordinary tables associated with the normal population distribution of Z for the 5 percent level if he were aware of the λ_3 and λ_4 of his population and was willing to assume an Edgeworth type distribution. In this connection the author notes that the Edgeworth restriction would not be serious if the size of sample was sufficient to enable one to neglect terms of order N^{-3}. If λ_3 and λ_4 are unknown one might use their sample estimate in the formulas to get an indication of the size of error involved.

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891[K].—N. F. GJEDDEBAEK, "Contribution to the study of grouped observations. Application of the method of maximum likelihood in case of normally distributed observations," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 135–159.

Let

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2), \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy, \qquad \varphi'(x) = \frac{d\varphi(x)}{dx},$$

$$z_1(x, y) = -\frac{\varphi(x+y) - \varphi(x)}{\Phi(x+y) - \Phi(x)}, \qquad z_2 = \frac{\varphi'(x+y) - \varphi'(x)}{\Phi(x+y) - \Phi(x)}.$$

Table 1, p. 151–154, contains 4D values of $z_1(x, y)$ for $y = 0(.1)4$ and $x = -.1h(.1)4$, where h is the largest integer $\leq 5y + 1$. Table 2, p. 155–158, contains 4D values of $z_2(x, y)$ for the same values of y and x in Table 1. Given y, both z_1 and z_2 are symmetrical about the value $x = -y/2$. Table 3, p. 159, contains 4D values of z_1 and z_2 for $y \to \infty$ and $x = -4.5(.1)0$. The functions z_1 and z_2 are used in an iterative procedure for determining the maximum likelihood estimates of the mean and standard deviation of a

normal population when the observations from this population have been grouped.

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892[K].—A. HALD, *The Decomposition of a Series of Observations Composed of a Trend, a Periodic Movement and a Stochastic Variable.* Copenhagen, G. E. C. Gads, 1948, 134 p., 17.2 × 25.1 cm. 8 kr.

The author defines polynomials $\xi_r(x_{ri})$, of degree $r = 1, \cdots, l$ ($\nu = 1, \cdots, n$; $i = 1, \cdots, k$), by means of the relations $\xi_{\mu\cdot} = 0$ (a dot in a subscript indicates a mean with respect to the corresponding variable), $\mu = 1, \cdots, l$, and

$$\sum_{\nu=1}^{n} \sum_{i=1}^{k} (\xi_{\tau\nu i} - \xi_{\tau\cdot i})(\xi_{\mu\nu k} - \xi_{\mu\cdot i}) = 0 \text{ for } \mu < \tau, \tau = 2, \cdots, l; \mu = 1, \cdots, l-1.$$

Here $\xi_{\tau\nu i}$ is used to denote $\xi_\tau(x_{\nu i})$.

The leading coefficients are assigned the value unity and then the remaining coefficients are determined by the defining equations. For $k = 1$ these become the ordinary CHEBYSHEV polynomials over equally spaced abscissas. The tables give values generally to 6S, of these polynomials of degrees 1 to 5 for the combinations $k = 3$, $n = 5, 7$; $k = 5$, $n = 5, 7, 8, 9, 10, 11$. For $k = 4$, $n = 25$ the degrees run from 1 to 4 and for $k = 12$, $n = 10$, the degrees run from 1 to 6. In addition explicit formulas for the polynomials tabulated are given (the author was unable to derive the general explicit formula) and the values, again generally to 6S, of the elements of the three matrices $\|\xi_{\tau\cdot i}\|$, $\|S_{\tau\mu}\|$, $\|S'_{\tau\mu}\|$, for the polynomials tabulated in which

$$S_{\tau\mu} = \sum_{\nu=1}^{n} \sum_{i=1}^{k} (\xi_{\tau\nu i} - \xi_{\tau\cdot i})(\xi_{\mu\nu i} - \xi_{\mu\cdot i}), \qquad (S_{\tau\mu} = 0, \tau \neq \mu),$$

and

$$S'_{\tau\mu} = \sum_{\nu=1}^{n} \sum_{i=1}^{k} \xi_{\tau\nu i}\xi_{\mu\nu i}.$$

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893[K].—H. O. HARTLEY, "The maximum F-ratio as a short-cut test for heterogeneity of variance," *Biometrika*, v. 37, 1950, p. 308–312.

The ratio of the largest to the smallest in a set of k variance estimates is proposed as a short-cut test in place of BARTLETT's test,[1]

$$M = N \log_e \{N^{-1} \sum_i \nu_i s_i^2\} - \sum_i \nu_i \log_e s_i^2$$

where the estimate s_i^2 is based on ν_i degrees of freedom and $N = \sum_i \nu_i$.

Since $\log_e s^2$ is approximately normal with variance $2/(\nu - 1)$,

$$F_{\max}(\alpha) = \exp \{w_k(\alpha)\sqrt{2/(\nu - 1)}\}$$

gives approximately the $100\alpha\%$ point of the ratio when the k estimates are based on the same number, ν, of d.f. and where $w_k(\alpha)$ is the corre-

sponding point of the distribution of the range in samples from normal (PEARSON & HARTLEY[2]). Tables 1 and 2 give comparisons of the approximate and exact 5% points of F_{max} to 3S for $k = 2$ and $\nu = 2(1)10, 12, 15, 20$, and for $\nu = 2$ and $k = 2(1)12$, respectively. The two sets of exact results were used to obtain second approximations by the adjustment, $F_{max}(\alpha)$ (second approx.) $= F_{max}(\alpha)$ (first approx.) $(1 + q_\nu q_k)$, with q_ν and q_k fitted to the exact values. The second approximations, with exact values in the first row and the first column, are given in Table 3, which shows upper 5% points of F_{max} to 3S in a set of k mean squares all based on ν d.f. for $k = 2(1)12$ and $\nu = 2(1)10, 12, 15, 20, 30, 60, \infty$.

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[1] M. S. BARTLETT, "Properties of sufficiency and statistical tests," R. Soc. London, *Proc.*, v. 160A, 1937, p. 268–282.
[2] E. S. PEARSON & H. O. HARTLEY, "The probability integral of the range in samples of n observations from a normal population," *Biometrika*, v. 32, 1942, p. 301–310.

894[K].—H. O. HARTLEY & E. S. PEARSON, "Tables of the χ^2-integral and of the cumulative Poisson distribution," *Biometrika*, v. 37, 1950, p. 313–325.

Let $$P(\chi^2, v) = 2^{-\frac{1}{2}v}\{\Gamma(\tfrac{1}{2}v)\}^{-1} \int_{\chi^2}^{\infty} e^{-\frac{1}{2}x} x^{\frac{1}{2}v-1} dx.$$

The authors have tabulated (p. 318–325) $P(\chi^2, v)$ to 5 places of decimals for $v = 1(1)20(2)70$ degrees of freedom and $\chi^2 = 0(.001).01(.01).1(.1)2(.2)10(.5)20(1)40(2)134$. At the same time the table provides the values of the cumulative Poisson distribution since $P(\chi^2, v) = \sum_{i=0}^{c-1} e^{-m} m^i/i!$ with $m = \chi^2/2$, $c = v/2$. Hence for $m \leqq 15$ the complete Poisson distribution is provided and for $m > 15$, only the truncated Poisson sum up to $c = 35$ ($v = 70$). Methods of interpolation in the table are thoroughly discussed. The authors point out that Pearson's table of the incomplete Γ-function[1] and MOLINA's table of the Poisson exponential limit[2] are more extensive, the former for χ^2 and the latter for the Poisson distribution. But Pearson's table involves a troublesome transformation and Molina's tables do not give $P(\chi^2, v)$ for odd values of v. In addition it may be noted that Salvosa's table of the Type III function, both areas, ordinates and the first six derivatives[3] may be used to evaluate $P(\chi^2, v)$ by setting t in SALVOSA's table $= (\chi^2 - v)(2v)^{-\frac{1}{2}}$, and $\alpha_3^2 = 8/v$. A comparison by the reviewer with the tables of $P(\chi^2, v)$ given in the Elements of Statistics by DAVIS & NELSON[4] for small values of χ^2 in a few instances revealed no divergencies, but for $v = 2$, $\chi^2 = 5$ Davis and Nelson give .08208 49986, while the present table gives .08209. The same phenomenon was noted for $v = 9$, $\chi^2 = 3.0$. No doubt the table will be found useful in statistical investigations.

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[1] KARL PEARSON, *Tables of the Incomplete Γ-Function*. London, 1922.
[2] E. C. MOLINA, *Poisson's Exponential Binomial Limit*. New York, 1945.

[3] L. R. SALVOSA, "Tables of Pearson's type III function," *Annals Math. Stat.*, v. 1, 1930, p. 191–198. Appendix, p. 1–187.
[4] H. T. DAVIS & W. F. C. NELSON, *Elements of Statistics with Applications to Economic Data*. Bloomington, Ind., 1935.

895[K].—J. M. HOWELL, "Errata to 'Control chart for largest and smallest values,'" *Annals Math. Stat.*, v. 21, 1950, p. 615–616.

In the paper cited in the title [*MTAC*, v. 4, p. 207–208] the author notes that numerical errors exist in Table I (as was pointed out by *MTAC*'s reviewer). Using values appearing in a paper by GODWIN,[1] HOWELL now gives a corrected version of Table I to 4D.

C. C. C.

[1] H. J. GODWIN, "Some low moments of order statistics," *Annals Math. Stat.*, v. 20, 1949, p. 279–285. [See *MTAC*, v. 4, p. 20.]

896[K].—R. F. LINK, "The sampling distribution of the ratio of two ranges from independent samples," *Annals Math. Stat.*, v. 21, 1950, p. 112–116.

Consideration is given to a sample of n ordered observations drawn from a population with variance σ^2. Let $w = (x_n - x_1)/\sigma$. Then for two samples of sizes n_1 and n_2 drawn from populations of the same variance, the ratio w_1/w_2 becomes the ratio of the two ranges. Table II gives values of R for all combinations of n_1 and $n_2 \leqq 10$ and for $\alpha = .005, .01, .025, .05$, and $.10$, such that

$$Pr\,(w_1/w_2 < R) = \alpha$$

where the samples are drawn from populations with the same variance. These values may be used for testing the hypothesis that two independent samples were drawn from normal populations with the same variance. This test is therefore comparable to the F test.

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897[K].—E. LORD, "Power of the modified t-test (u-test) based on range," *Biometrika*, v. 37, 1950, p. 64–77.

For a sample of n independent observations from a normal population with unknown standard deviation, σ, the u-test is the ratio of the normal deviate to an independent range estimate of the standard error. The power of a two-tailed test, $\beta(\rho)$, against an alternative $\rho = \xi/\sigma$, is the sum of the two power components, $\beta'(\rho) = Pr\,\{\text{lower tail}|\rho\}$ and $\beta''(\rho) = Pr\,\{\text{upper tail}|\rho\}$. The standardized error of NEYMAN & TOKARSKA[1] is ρ_α such that $1 - \beta(\rho_\alpha) = \alpha$ for any significance level α. Six tables are given to compare the standardized errors of the u-test and the t-test for $\alpha = .05$ and $\alpha = .01$.

Table I, p. 67, shows to 5D the power components $\beta'(\rho)$ for $\rho = 1(1)10$ and $\beta''(\rho)$ for $\rho = 1(1)3$ and $\nu = 1(1)20$, ∞ of the t-test, where ν is the number of degrees of freedom for estimating variance. Table II, p. 68, gives the standardized errors for the t-test, $\rho_{.05}$ to 3D and $\rho_{.01}$ to 2D for $\nu = 1(1)20, 30, 60, 120, \infty$.

Table III, p. 70, gives the power components for the u-test to 5D for $n = 2(1)20, \infty$; $\beta'(\rho)$ for $\rho = 1(1)10$ and $\beta''(\rho)$ for $\rho = 1(1)3$. Table IV,

p. 71, shows the standardized errors for the u-test, $\rho_{.05}$ to 3D and $\rho_{.01}$ to 2D for $n = 2(1)20, \infty$.

In Table V, p. 72, there are given values of $\rho_{.05}$ to 2D for the u-test based on the average range in m samples of n independent observations each, for $n = 2(1)20$ and $m = 1(1)10(5)20, 30, 60, 120$. From Table V, it appears that the optimum grouping of $N (= mn)$ observations is one such that $6 \leqq n \leqq 10$ in general.

Table VI, p. 76, gives the ratio of the standardized errors at 5% and 1% levels of the u-test to the t-test for 3D for $n = 2(1)20, \infty$. The ratio is $\leqq 1.024$ for the 5% level and $\leqq 1.045^{+}$ for the 1% level.

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[1] J. NEYMAN & B. TOKARSKA, "Errors of the second kind in testing 'Student's' hypothesis," Am. Stat. Assn. *Jn.*, v. 31, 1936, p. 318–326.

898[K].—P. B. PATNAIK, "The use of mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.

Let $x_1, x_2, \cdots, x_n$ be a random sample of n observations, arranged in ascending order of magnitude, drawn from a normal population with mean μ and variance σ^2. Define the range by $w_n = x_n - x_1$, and if there are m such independent samples with n observations each, the mean range by $\bar{w}_{m,n}$. Further d_n is defined by $E(w_n) = \sigma d_n$. The author approximates the distribution of $\bar{w}_{m,n}$ by means of the first two moments of $\bar{w}_{m,n}$ and the χ^2 distribution with v degrees of freedom and scale factor c. In Table 1, needed for the distribution of $\bar{w}_{m,n}$, are given the values of d_n for $n = 3(1)10$ to 4D, v (the degrees of freedom which are not integers) to 3D, c, the scale factor to 4D, for $m = 1(1)5$, and $n = 3(1)10$. Table 2 gives the comparisons of the exact and approximate percentage points of $u = d_n x/\bar{w}_{m,n}$ for $n = 5, 8, 10$, $m = 1, 3, 5, 10$, $\alpha = .1, .05, .02, .01$, where x is distributed normally with mean zero, and variance σ^2 and α denotes the percentage points. The exact results are from LORD's computations,[1] while the approximate results are based on the author's approximation to the u distribution. Table 3 compares the exact power function of u with the author's approximate Student's non-central t, for $\mu/\sigma = 1, 2, 3$, $\mu \neq 0$, $n = 5, 10$, $m = 3, 5$ for a one-tailed region, $\alpha = .05$, to 3D. For the symmetrical or double-tailed test, comparisons of the power are made for $m = 1$, $n = 10$; $m = 2$, $n = 10$; $m = 4$, $n = 5$; $m = 5$, $n = 3$, and $m = 2$, $n = 5$ usually to 3D. The results are particularly useful in quality control and wherever it is desirable to reduce computation time by use of the mean range in place of the sample standard deviation as an estimator of the population standard deviation.

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[1] E. LORD, "The use of range in place of standard deviation in the t-test," *Biometrika*, v. 34, 1947, p. 41–67; "Power of the modified t-test (u-test) based on the range," *Biometrika*, v. 37, 1950, p. 64–77. [RMT 897]

899[K].—L. W. POLLAK, assisted by U. N. EGAN, *All Term Guide for Harmonic Analysis and Synthesis Using 3 to 24; 26, 28, 30, 34, 36, 42, 44, 46, 52, 60, 68, 76, 84 and 92 Equidistant Values.* (Department of Industry and Commerce, Meteorological Service, *Geophysical Publications*, v. 2.) Dublin, Stationery Office, 1949, xx, 185 p. 24.5 × 30.2 cm. £2 s2.

The purpose of the present work is to facilitate the work of fitting the Fourier series

$$y = p_0 + p_1 \cos x + p_2 \cos 2x + \cdots + q_1 \sin x + q_2 \sin 2x + \cdots$$

to a set of n equidistant values, where n is any one of the numbers given above in the title.

The tables are supplementary to those published by the author in 1947 and described in *MTAC*, v. 2, p. 306. The tables of the "Schedule," which comprises most of the work (185 p.), give the values to 5D of the functions

$$\cos \frac{2\pi km}{n}, \qquad \sin \frac{2\pi km}{n},$$

where n is any one of the numbers given above in the title, $m < n$, and $k \leq \frac{1}{2}n$.

In a typical entry n is printed in large blackface type, let us say $n = 20$, and for this number there are then 10 tables corresponding to $k = 1$ to $10 = \frac{1}{2}n$. The value of k is printed in smaller blackface type at the top of each table. Two arguments are given, one for m, denoted by i by the author, which ranges over the positive integers, and the other for $i' = km - sn$, where s is chosen so that $km - sn$ is positive and less than n.

Two other entries are included, which refer to tables in the author's *Rechentafeln zur harmonischen Analyse*, Leipzig, 1926, by means of which the multiplication of the harmonic terms is facilitated. The choice of n given above was guided by the choice of values found in the *Rechentafeln*.

The present work is a valuable addition to other similar tables which have been computed and which are listed in *MTAC*, v. 1, p. 193.

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900[K].—E. M. SCHROCK, *Quality Control and Statistical Methods.* New York, Reinhold Publishing Corporation, 1950. xi + 213 p., 17.1 × 24.1 cm. $5.00.

Tables 9.7a, p. 69–71, and 9.7b, p. 73–122, of this book were prepared with the apparent purpose of routinizing the construction of control charts for industrial applications covering most situations which are likely to be encountered in practice, in so far as problems dealing with the per cent defective are concerned.

Table 9.7a labeled, "Table of Width of 3-Sigma Band in Per Cent," gives the amounts to be added to and subtracted from the average per cent defective or process level, $\bar{p}$, in order to obtain 3-sigma upper and lower control limits for sample per cent defectives. Thus Table 9.7a gives values of $3[\bar{p}(1 - \bar{p})/N]^{\frac{1}{2}}$ to 4D, where N is the sample size, for $N = 10(10)100$-

(100)4000(200)5000(500)10000, and $\bar{p}$ = .001, .0025, .005, .01, .015, .02(.01)-.1(.02).2(.05).5. For non-tabulated $\bar{p}$'s and N's, the author states, "although the relationships are curvilinear, straight line interpolation between rows and columns may be used for all practical purposes."

Table 9.7b, Section One, gives 3-sigma upper and lower control limits for the number of defectives to the nearest integer and for the per cent defective to 2D and also gives the expected number of defectives for ranges of $\bar{p}$ and N as above. GRANT[1] gives a comparable table of upper and lower control limits alone. Section II of this table is a rearrangement of the same information for numbers of defectives, giving separately for each N = 100-(100)4000(200)5000(500)10000, the values for the same $\bar{p}$'s as above up to .35. Thus Section One of Table 9.7b with different sample sizes on each page is convenient for testing control with respect to an established or standard process level of per cent defectives, whereas Section II of Table 9.7b "is extended for use in situations where several different quality characteristics, each with a different quality level, will be checked on samples of the same size."

Tables 9.7a and 9.7b occupy some 54 pages of the book and because of the wide ranges of sample sizes and values of per cent defective should be of considerable value as a ready reference to the quality control engineer for industrial problems involving per cent defective. Figures 9.7a and 9.7b are included in the book for obtaining limits for sample sizes other than those given in Tables 9.7a and 9.7b or for values of per cent defective larger than 35%. These figures can also be used for 3-sigma confidence intervals for the per cent defective of the lot or material samples. Examples are given for the use of Tables 9.7a, 9.7b and Figures 9.7a, 9.7b.

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[1] E. L. GRANT, *Statistical Quality Control*. New York, 1946.

901[K].—C. A. WILLIAMS, JR., "On the choice of the number and width of classes for the chi-square test of goodness of fit," Am. Stat. Assn. *Jn.*, v. 45, 1950, p. 77–86.

This paper contains a presentation and discussion, intended mainly for the non-mathematical user of tests of goodness of fit, of results of a paper by MANN and WALD[1] which make it possible to optimize, in a certain sense, the number of classes and the choice of class-boundaries for the chi-square test. Tables are given showing, for the 1% and the 5% significance level and the sample sizes N = 200(50)1000(100)1500, 2000, the optimum number k of classes, as well as some information related to the power of the test. The author suggests that the tabulated values of k may be halved with little loss of power.

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[1] H. B. MANN & A. WALD, "On the choice of the number of class intervals in the application of the chi-square test," *Annals Math. Stat.*, v. 13, 1942, p. 306–317.

902[L].—Akademiĭa Nauk, SSSR. Matematicheskiĭ Institut im. V. A. Steklova, Institut Mekhaniki, *Tablitsy znacheniĭ Funktsiĭ Besselia ot mnimogo Argumenta*. Edited by I. M. Vinogradov & N. G. Chetaev, Moscow, Leningrad, 1950, xii, 404 p. 39 roubles.

Here we have finally the work which was started in 1930 and ready for publication in 1941 [*MTAC*, v. 3, p. 66 and v. 5, p. 112], after collaboration of various individuals and academies, as set forth in the preface. The world war delayed publication, but after further revision by the present editors, an edition of 2,500 copies was finally printed last year.

After the introductory matter there are two sections (p. 1–201, 203–403) each containing four tables. These tables are of (1) $I_0(x)$ to 8D; (2) $I_1(x)$ to 9D; (3) $2\pi^{-1}K_0(x)$ and (4) $2\pi^{-1}K_1(x)$ to 8S or 8–12D; (5) $I_{\frac{1}{2}}(x)$ and (6) $I_{-\frac{1}{2}}(x)$ to 8S; (7) $K_0(x)$ and (8) $K_1(x)$ to 8S or 8–12D; all for $x = 0(.001)10, \Delta$.

The only publications referred to in the literature list are: Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., 1944 and the Russian translation of 1949. Gray-Mathews-MacRobert, *A Treatise on Bessel Functions*. The date given is 1931, but presumably 1936 was intended. *A Guide to Tables of Bessel Functions* as in *MTAC*, v. 1, no. 7, 1944. FMR, *Index*, 1946. And finally there is a reference to tables of $I_0(x)$ and $I_1(x)$ in BAAS, *Report*, 1893 and 1896, but these tables are attributed to Aldis, and not to the real author A. Lodge; these 9D tables were for $x = 0(.001)$-5.1, Δ. This mention of the name Aldis, where there was no reason for it, and general considerations, suggest very definitely that other unmentioned important tables were well known to the compilers and computers. Nevertheless, it is undoubtedly true that most of the values given in the volume before us are new.

Let us recall just what was available before, apart from the tables of Lodge. In R. Soc. London, *Proc.*, v. 64, 1899, p. 218–223, Aldis gave tables of $I_0(x)$, $I_1(x)$, for $x = [.1(.1)6; 21D]$, $[6(1)11; 18D]$. BAASMTC, *Math. Tables*, v. 6, 1937, contained tables of $I_0(x)$, $I_1(x)$, for $x = [0(.001)4; 8D]$, $[4(.001)5; 7D]$, δ^2 and of $K_0(x)$, $K_1(x)$, for $x = [0(.01)2; 8D]$, $[2(.01)4; 9D]$, $[4(.01)5; 10D]$, δ^2. In the case of I_0, I_1 it was found that the Lodge tables were not dependable in final digits, and hence the calculations were derived by subtabulation in the 21D values of Aldis. In NBSMTP, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, 1943, second edition 1947, there are tables of $I_0(x)$, $I_1(x)$, for $x = [0(.01)10; 10D]$. These tables occur again in NBSCL, *Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments*, New York, 1950, published almost simultaneously with the volume under review. In this volume for $x = [0(.01)10; 10D]$ are also tables of (3) and (4) $2\pi^{-1}K_0(x)$, $2\pi^{-1}K_1(x)$. The only previous tables of $I_{\pm\frac{1}{2}}(x)$ are those of A. N. Dinnik, 1915 [*MTAC*, v. 1, p. 287], $I_{\frac{1}{2}}(x)$, for $x = [0(.1)15; 4$–$6S]$; $I_{-\frac{1}{2}}(x)$, for $x = [0(.1)7; 5$–$6S]$. But Dinnik's other tables have been found notoriously unreliable [*MTAC*, v. 2, p. 379]. Finally there is NBSAMS 1, *Tables of the Bessel Functions*, $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, 1948 [*MTAC*, v. 3, p. 187–188], $K_0(x)$ and $K_1(x)$ for $x = [0$-$(.0001).033(.001)1; 7S]$, Δ^2.

From these statements the exact relation of the Russian volume to what had appeared before (which excludes NBSCL, 1950) is clear. The only exact duplication is in $I_0(x)$, $I_1(x)$, for $x = 0(.001)4$ in the BAASMTC volume, and in the Lodge $I_1(x)$ table.

Thus, if the tables in this new volume are found to be accurate,[1] a notable addition has been made to the large amount of tabular material dealing with Bessel functions.

NOTE ADDED IN PROOF: A friend (who desires that his name be withheld) has drawn my attention to ten additional errata, each of 3 to 10000 units, in final decimal places:

Function	x	for	read
$I_0(x)$	1.009	175	075
	2.46	8582	7582
$I_1(x)$	4.06	6	3
	4.67	6	3
	4.87	4	1
	4.88	5	2
	5.07	3	0
	5.08	6	3
$I_0(x)$	6.79	9981	1981
	9.73	46664	56664

Because of these, and many other less serious errors already observed these Russian tables of the Academy of Science must be characterized as decidedly unreliable.

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[1] On spot-checking the 9D table of $I_1(x)$ in the work under review with the 10D table of NBSMTP, 1943, one difference of 4 units in the end-figure was found at $x = .5$, and 49 differences of one or two units in end-figures were found for the following values of x: .03, .05, .07, .09, .7, .75, .8, 1.02, 1.03, 1.11, 1.15, 1.16, 1.17, 1.18, 1.25, 1.3, 1.35, 1.36, 1.37, 1.39, 1.5, 1.55, 1.57, 1.59, 1.7, 1.85, 1.87, 1.94, 1.97, 1.98, 2.67, 2.68, 2.69, 2.7, 2.85, 3.03, 3.4, 3.95, 4.35, 4.55, 4.65, 4.71, 4.72, 4.76, 4.77, 4.79, 4.85, 5.05.

At every one of these 50 places, except for $x = .5$ (because of the very weakly printed left part of Lodge's ending 4, it was copied in the Russian table as a 1), the values of the Russian table were identical with the values of the Lodge (1893) 9D table of $I_1(x)$. Hence, it seems evident that the Lodge table was copied completely without any checking whatever. Since the British Committee has told us that the 9D tables of $I_0(x)$ and $I_1(x)$ "were subject to errors of two or three units in the last decimal, so that rounding off would not yield 8-figure values of the standard of accuracy set for the Committee's tables," we can assert that the Russian tables are probably incorrect at each of the places indicated above. Furthermore, spot-checking the 8D table of $I_0(x)$, differences were found at $x = .07$, 3.3, 4.4, where there was evident rounding off of Lodge values. Since correct 8 D values are given in the BAASMTC tables it is definitely suggested that the Russians also copied their table of $I_0(x)$, up to $x = 5.1$, rounded off from Lodge, and that they did not have the $I_0(x)$ table in BAASMTC. Since the Lodge table of $I_1(x)$ abridged to interval .01 was reprinted in GRAY & MATHEWS, 1895, 1922, 1936, all the probable errors noted above are also to be found in G. & M. and G. M. & Mac R.

In a brief spot checking beyond $x = 5.1$ up to $x = 7$, unit differences in end figures were found at $x = 5.3$, 5.45, 5.57, 6.15, 6.35, 6.45, 6.75.

903[L].—P. A. CARRUS & C. G. TREUENFELS, "Tables of roots of incomplete integrals of associated Legendre functions of fractional orders," *Jn. Math. Phys.*, v. 29, 1951, p. 292–299.

$P_\mu^1(\cos\theta)$ is the associated Legendre function of order one and fractional index μ. Tabulated are the zeros of

$$P_\mu^1(\cos\theta) = 0, \qquad \frac{d}{d\theta} P_\nu^1(\cos\theta) = 0$$

treated as a function of the index.

Table I gives the first 50 values of μ for $\theta = 90°(5°)175°$, while Table II gives the first 50 values of ν for $\theta = 90°(5°)130°$. The k-th root is denoted by μ_k, ν_k. In both tables entries are mostly to 5S. Values were derived from power series, for the smaller roots, and from asymptotic expansions due to PAL[1] for the larger roots. Convergence difficulties prevented extension of Table II beyond 130°.

No indication as to accuracy of the tables is given, though it is stated that interpolation of the roots θ-wise is practicable with five point Lagrangean interpolation. The reviewer has spot checked the tables by differencing and in a number of cases found that third differences are not smooth, indicating that some errors are present. For example, in Table I, entry for $k = 1$, $\theta = 130°$ is undoubtedly in error. The entry 1.3001 should probably read 1.2975. In Table II, there is a typographical error for the entry $k = 49$, $\theta = 90°$: *for* 88.000 *read* 98.000. In Table II, for k large, third differences are not smooth. The reviewer has not differenced all the entries. In absence of error, interpolation with four point interpolation should be sufficient to insure full accuracy of Tables I and II.

In Tables III and IV, values of the integral $\int_{x_0}^{0}\{P_n^1(x)\}^2 dx$ are presented. Here $x = \cos\theta$, n stands for μ or ν, and tabulations are given for each value of the index found in Tables I and II. However, in Table III, $\theta = 90°(5°)145°$, in Table IV, $\theta = 90°(5°)130°$. Entries are to 4S and 5S. Finally, Table V gives values of $\int_0^1\{P_n(x)\}^2 dx$ to 5D for $n = 0(0.05)1.0$ together with second differences.

All tables are new, though PAL[1] had previously determined some μ and ν values of the functions considered for various orders and values of θ in the first quadrant.

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[1] B. PAL, "On the numerical calculation of the roots of the equations $P_n^m(\mu) = 0$ and $\frac{d}{d\mu}P_n^m(\mu) = 0$ regarded as equations in n," Calcutta Math. Soc., *Bull.*, v. 9, 1918, p. 85–95 and v. 10, 1919, p. 187–194.

904[L].—A. H. CARTER & O. A. WILLIAMS, JR., "A new expansion for the velocity potential of a piston source," Acoustical Soc. Amer. *Jn.*, v. 23, 1951, p. 179–184.

A small table is given (p. 182) of the values of the hypergeometric function $F(-(m-1), -m; 1; b^2)$ which the authors denote by f_{2m}. These functions are actually polynomials of degree $m-1$ in b^2. Thus

$$f_4 = 1 + 2b^2, \qquad f_6 = 1 + 6b^2 + 3b^4.$$

The 6D tables give $f_{2m}(b)$ for

$$m = 1(1)8 \quad \text{and} \quad b = 0(.1)1.$$

There is also a table of the coefficients of f_{2m} for $m = 1(1)8$. These functions occur as factors of coefficients in the expansion of the velocity potential of a piston source in terms of HANKEL functions.

D. H. L.

905[L, P].—R. A. CLARK, "On the theory of thin elastic toroidal shells," *Jn. Math. Physics*, v. 29, 1950, p. 146–178.

Table I, p. 156, gives 4D values of

$$\int_0^\phi \{\sin t/(1 + \lambda \sin t)\}^{\frac{1}{2}} dt$$

for $\lambda = 0(.1).5$ and $\phi = 0^0(2^0)90^0$.

Table II, p. 167, gives 3D values of the real and imaginary parts of $T(x)$ and $dT(x)/dx$ for $x = 0(.05)5$ where $T(x)$ is that solution of

$$\frac{d^2T}{dx^2} - ixT = 1$$

which is asymptotic to i/x for large (positive) x. In terms of LOMMEL's function

$$T(x) = -\tfrac{2}{3}(ix)^{\frac{1}{2}} S_{0,\frac{1}{3}}[\tfrac{2}{3}(ix)^{\frac{3}{2}}].$$

A. E.

906[L].—R. A. FISHER, "Gene frequencies in a cline determined by selection and diffusion," *Biometrics*, v. 6, 1950, p. 353–361.

This paper contains a table of the solution $q(x)$ of the non-linear differential equation $q'' = 4xq(1 - q)$ subject to the two point boundary condition $q(0) = \frac{1}{2}$, $q(\infty) = 0$. The table gives $q(x)$ to 8D for $x = 0(.02)5.7$. Beyond this range $q(x) < 10^{-8}$. The table was computed on the EDSAC [*MTAC*, v. 4, p. 61–65].

D. H. L.

907[L].—CARL-ERIK FRÖBERG, "On determination of proton-proton interaction from scattering experiments," *Arkiv för Fysik*, v. 3, 1951, p. 1–23.

The COULOMB wave equation containing an additional nuclear interaction potential $V(r)$ is

$$d^2y/dr^2 + [k^2 - 2\alpha kr^{-1} - l(l + 1)r^{-2}]\, y = -V(r)y. \tag{1}$$

If $V(r) \equiv 0$ a system of two linearly independent solutions can be given by u_k and v_k, such that $u_k(0) = 0$, $u_k \sim \sin(kr + \eta)$ as $r \to \infty$ and $v_k \sim \cos(kr + \eta)$ as $r \to \infty$, with $\eta = -\alpha \log(2kr) - \frac{1}{2}l\pi + \arg\Gamma(i\alpha + l + 1)$. If $V(r)$ is different from zero an additional phase-shift $\delta(k)$ appears in the asymptotic expression for the regular solution y of (1), so that $y(0) = 0$ and $y \sim \sin[kr + \eta + \delta(k)]$ as $r \to \infty$. The author discusses in this paper the determination of $V(r)$ if $\delta(k)$ has been determined experimentally for every positive k. The relation between V and δ is given by

$$V(r) = -\frac{2}{\pi}(2\alpha k)^2 \int_0^\infty \alpha^{-3} \sin\delta(\alpha)\, \varphi(\alpha, \lambda)\, d\alpha \tag{2}$$

where

$$\varphi(\alpha, \lambda) = \left(\frac{d}{d\rho} u_k(\rho) v_k(\rho)\right)_{\rho = \lambda/(2\alpha)}$$

and

$$\lambda = 2\alpha kr.$$

To determine $V(r)$, a knowledge of the function $\varphi(\alpha, \lambda)$ is necessary. Therefore the author gives asymptotic expansions for u and v. The main part of the paper consists of two numerical tables. The first table is for $v_1(\rho)$ and $v_1'(\rho)$ with $l = 0$ for $\alpha = .01$ and $0 \leqslant \rho \leqslant 6$, $\alpha = .02$, $.03$ and $0 \leqslant \rho \leqslant 4$, $\alpha = .04$ and $0 \leqslant \rho \leqslant 2$, $\alpha = .06(.02).12(.04).2(.05).3(.1)1$ and $0 \leqslant \rho \leqslant 1$: the intervals in ρ vary. The second table is for $\varphi(\alpha, \lambda)$, α ranging from .01 to ∞ and λ from .012 to .120, with various intervals. The last section of the paper gives the computation of $V(r)$ for two cases, one where $\sin \delta = \text{const.}\ \alpha^2$ and $\alpha = .02$, the other where $\sin \delta$ is approximately constant (.79) and $.006 < \alpha < .02$. Three smaller numerical tables are included. The tables were computed on the Swedish automatic computing machine BARK in Stockholm [*MTAC*, v. 5, p. 29–34].

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908[L].—R. GRAMMEL, "Tafeln der verallgemeinerten Kreisfunktionen Sin(4)v, Cos(4)v, Sin(6)v, Cos(6)v, Sin(8)v, Cos(8)v," *Ing.-Arch.*, v. 18, 1950, p. 251–254.

The functions $x = \cos(n)v$ and $y = \sin(n)v$ have been defined in an earlier paper[1] as the inversion of the integrals

$$v = \int_x^1 f_n(t)\,dt = \int_0^y f_n(t)\,dt \tag{1}$$

where

$$f_n(t) = (1 - t^n)^{(1-n)/n}.$$

For $n = 2$ these functions reduce to trigonometric functions. For any even integer n, they provide a parametric representation of the curve

$$x^n + y^n = 1, \tag{2}$$

and they have certain periodic and symmetry properties with half-period

$$\pi_n = \int_0^1 f_n(t)\,dt.$$

In the present paper 6D numerical tables of $\cos(n)v$ and $\sin(n)v$ are given for $n = 4, 6, 8$ and $v = 0(.01)1(.1)4$ and for $2v/\pi_n = 0(.05)1$. The earlier paper[1] gives 3D values of π_n for $n = 2(1)10$, ∞, and the present paper 6D values of $\frac{1}{4}\pi_n$ for $n = 4, 6, 8$. For $n > 2$ one has $\pi < \pi_n \leq 4$.

The infinite series of the earlier paper[1] have been used for the computation of the functions for $0 < v \leq \frac{1}{4}\pi_n$, symmetry and periodic properties combined with interpolation, for $v > \frac{1}{4}\pi_n$. The original computations were carried to 7 or 8D, and the author states that all six decimals printed are correct.

Differencing of the tables was used as a check, the relation (2) was checked for every fifth entry, and for $n = 4$ the connection with Jacobian elliptic functions was used to check the tables.

The computations were carried out by F. JINDRA.

A. E.

[1] R. GRAMMEL, "Eine Verallgemeinerung der Kreis- und Hyperbelfunktionen." *Ing.-Arch.*, v. 16, 1948, p. 188–200.

909[L].—E. KREYSZIG, "Über den allgemeinen Integralsinus Si(z, α)," *Acta Math.*, v. 85, 1951, p. 117–181.

The function

$$\text{Si}\,(z, \mu) = \int_0^z t^{-\mu} \sin t\, dt \tag{1}$$

was introduced by A. WALTHER.[1] It is related to the incomplete gamma function in the same manner as the ordinary sine integral, Si (z, 1), is related to the exponential integral function. Together with the generalized cosine integral,

$$\text{Ci}\,(z, \mu) = \int_0^z t^{-\mu} \cos t\, dt, \tag{2}$$

the integrals investigated in this paper account for the incomplete gamma function, sine-, cosine-, and exponential integrals, and Fresnel's integrals.

The author investigates Si and Ci in the complex domain. $z = x + iy$, $\mu = \alpha + i\beta$, $0 < \alpha < 2$ in (1), and $0 < \alpha < 1$ in (2). In the theoretical part he discusses power series expansions in ascending powers of z, the connection with other functions including confluent hypergeometric functions, asymptotic expansions as $z \to \infty$, and a convergent expansion in incomplete gamma functions. He investigates the zeros and the product representation of the functions in question and comments on the numerical computation of the zeros.

Table 1 (p. 155–6) gives 3D values of Si (x, α) for $x = 0(.2)4(.5)20$, $\alpha = .25(.25)1.75$.

Table 2 (p. 157–163) gives 2D (sometimes 3S) values of the real and imaginary parts of Si($x + iy$, α) for $x = 0(1)20$, $y = 0(1)5$, $\alpha = .25(.25)1.75$.

Table 3 (p. 164) gives 2D values for the real and imaginary parts of the first three zeros of Si (z, α) for $\alpha = 0(.25)1.75$.

Table 4 (p. 165) gives 5D values of Si (∞, α) for $\alpha = .05(.05)1.95, 1.99, 1.995, 1.999, 2$, and 5D values of Ci ($\infty$, α) for $\alpha = .05(.05)1$.

Table 5 (p. 166) gives 8D values of

$$P(n + 1, z) = \int_0^z e^{-t} t^n\, dt$$

for $z = 1, -1, i$ (in the last case real and imaginary parts) and $n = 0(1)10$.

There are relief diagrams and charts illustrating the functions tabulated, other diagrams; and an extensive bibliography of papers on both theory and applications, and also of numerical tables of the sine integral and FRESNEL's integral.

The tables are said to be accurate to within a unit of the last decimal.

Although the relation

$$\text{Ci}\,(z, \mu) = z^{-\mu} \sin z - \mu\, \text{Si}\,(z, \mu + 1), \qquad 0 < \alpha < 1,$$

allows one to compute Ci from Si, tables of Ci (z, μ) would be a useful addition. Together with the tables of the present paper they could serve to compute the incomplete gamma function in the complex domain.

A. E.

[1] A. WALTHER, "Anschauliches zur Gibbsschen Erscheinung und zur Annäherung durch arithmetische Mittel," *Math. Zeit.*, v. 42, 1937, p. 355–364.

910[L].—L. M. MILNE-THOMSON, *Jacobian Elliptic Function Tables*. Dover Publications Inc., 1950. xii, 132 p., 12.7 × 18.9 cm. $2.45.

The purpose of this book is explained in the opening sentence of its preface. "The widespread belief that calculations involving elliptic functions are difficult is due not to the nature of the calculations themselves but to the lack of suitable numerical tables wherewith to perform them." In fact, a glance at A. FLETCHER'S "Guide to Tables of Elliptic Functions" [*MTAC*, v. 3, p. 229–281] suffices to show that while there is a considerable body of numerical material for the use of a professional computer, and also for some special purposes, yet there is no handy and easily available book of tables which could be used by the general practitioner of the various sciences as easily, and as safely as, say, a table of logarithms. The present book attempts to fill this gap.

The standard notations are used, except that the parameter $m = k^2$ is used rather than the modulus k, $\operatorname{sn}(u|m)$ is written for $\operatorname{sn}(u, k)$, and similarly for cn, dn. The *complementary parameter* is $m_1 = 1 - m$.

The first 39 pages are devoted to the definition of Jacobian elliptic functions and a description of their principal properties, examples showing the use of the tables, complete elliptic integrals, Fourier series and power series expansions of elliptic functions, periods, zeros, poles and residues, formulas for special values of the argument, change of argument (including double and half arguments and addition theorems), change of parameter (transformations), approximations, complex arguments, differentiation, WEIERSTRASS'S $\wp$-function, integrals of Jacobian elliptic functions, elliptic integrals and the zeta function, conformal mapping, factorisation of cubic and quartic polynomials, and application to the pendulum problem. In brief, these 39 pages give practically all the information constantly needed when using elliptic functions.

The following tables and graphs form the main body of the book under review.

$\operatorname{sn}(u|m)$. Graphs for $m = 0$, $\frac{1}{2}$, 1 and $0 \leq u \leq 2\pi$ (p. 41). 5D tables with first differences for $m = 0(.1).5$ and $u = 0(.01)2$, $m = .6$, $.7$, $.8$ and $u = 0(.01)2.5$, $m = .9$ and $u = 0(.01)3$, and $m = 1$ and $u = 0(.01)3(.1)6.5$, with values of the quarter-period K at the foot of each column (p. 42–61).

$\operatorname{cn}(u|m)$. Graph (p. 63) and tables (p. 64–83) in the same range as for sn, except that for technical reasons the values for $m = 1$ (which are common to cn and dn) and $u \geq 3.5$ are distributed over various pages of the dn-table.

$\operatorname{dn}(u|m)$. Graph (p. 85) and tables (p. 86–105) again for the same values of m and u as for sn, except that $\operatorname{dn}(u|0) = 1$ for all u and need not be given, and the column thus saved is used to accommodate tables for $\operatorname{cn}(10u|1) = \operatorname{dn}(10u|1)$ up to the point ($u = 1.29$) where these functions vanish (to 5D accuracy).

Complete elliptic integrals and the nome (p. 106–109). 7D tables of K, K', E, E', and 8D tables of $q = \exp(-\pi K'/K)$, and $q_1 = \exp(-\pi K/K')$ for $m = 0(.01).5$, with values of m_1 printed on the right margin and an arrangement like that of trigonometric tables to enable the user to read off values for $.5 \leq m \leq 1$.

$Z(u)$. 7D table (p. 112–123) for $m = 0(.1).6$ and $u = 0(.01)2$, $m = .7$, $.8$ and $u = 0(.01)2.5$, and $m = .9$, 1 and $u = 0(.01)3$, all with Δ''.

The book is very well printed, and the legibility of the tables is excellent, except that the table for $Z(u)$ seems to have been reproduced photographically from the *Proceedings* of the Royal Society of Edinburgh (v. 52, 1931, p. 239–250) and the considerable reduction in size makes for small figures and a somewhat crowded page.

A. E.

911[L, V].—M. MUNK & G. RAWLING, *Tables of Chaplygin Functions*. Naval Ordnance Laboratory Memorandum 10816, 57 pp. (1950). [Available to government contractors only.]

This paper consists of three tables of certain functions designated by the generic title of CHAPLYGIN functions, which arise as particular solutions of Chaplygin's differential equation. They are closely related to the physical plane stream function or to the LEGENDRE reciprocal potential of the hodograph plane in the computation of plane two-dimensional compressible flows. The adiabatic index is taken as 1.4.

This differential equation is:

$$\tau(\tau - 1)\frac{d^2Z_m}{d\tau^2} + [(m + 1 + \beta)\tau - (m + 1)]\frac{dZ_m}{d\tau} - \tfrac{1}{2}\beta m(m - 1)Z_m = 0$$

in which $\beta = (\gamma - 1)^{-1} = 5/2$, and τ is a dimensionless speed variable related to the Mach number by $\tau = M^2(2\beta + M^2)^{-1}$ and to the reduced velocity w by $\tau = w^2$.

Particular solutions tabulated are $Z_m = F(a_m, b_m; c_m, \tau)$, where $a_m + b_m = m + \beta$, $a_m b_m = -\tfrac{1}{2}\beta m(m - 1)$, $c_m = m + 1$; and $\bar{Z}_{-m}$, where $\bar{Z}_{-m}$ denotes a second independent solution; *not negative values of the parameter* m. This second solution $\bar{Z}_{-m}$ contains a logarithmic term and has been normalized to approach unity as w tends to zero.

The other functions were computed from the relations

$$L_m = w^m Z_m \qquad \bar{L}_{-m} = w^m \bar{Z}_{-m},$$
$$\psi_m = \rho(mL_m - m^{-1}wL'_m), \qquad \bar{\psi}_{-m} = \rho(m\bar{L}_{-m} - m^{-1}w\bar{L}'_{-m})$$

where $\rho = (1 - w^2)^{5/2}$, and a prime denotes d/dw.

Values of the functions for $w = 6^{-1}$, sonic speed, were found by interpolation using NEWTON's backward formula.

The tables give 6S values of the functions

$$Z_m, Z'_m, \bar{Z}_{-m}, \bar{Z}'_{-m}, L_m, L'_m, \bar{L}_{-m}, \bar{L}'_{-m}, \psi_m, \bar{\psi}_{-m}$$

for $m = 2(1)10$ and $w = .01(.01).5$ and for $w = 6^{-1}$ to which latter value of w a separate table is devoted.

The tables were produced by the computational facilities of the Naval Ordnance Laboratory.

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912[L].—NBSCL, *Table of the Bessel Functions* $Y_0(z)$ *and* $Y_1(z)$ *for Complex Arguments*. Columbia Univ. Press, 1950. xl, 427 p. 20 × 26.5 cm. $7.50.

This volume maintains the high standards of the earlier tables, prepared by the computation group under A. N. LOWAN. It presents values

of the real and imaginary parts of the second solutions Y_0 and Y_1 of the Bessel equation, according to the magnitude ρ of the argument $\rho e^{i\varphi}$ over the range [0(0.01)10; 10D] and according to the phase angle φ over the range [0°(5°)90°]. Auxiliary tables of $Y_0 - (2/\pi)J_0 \log \rho$ and $Y_1 - (2/\pi)J_1 \log \rho + (2/\pi\rho)e^{-i\varphi}$, over the range [$\rho$ = 0(0.01)0.50; φ = 0°-(5°)90°; 10D], are provided, to facilitate interpolation for small values of ρ. In addition, the position of some of the zeros of $Y_0 = Y_1'$ and of Y_1 and Y_0' off the real axis are given, together with values of Y_0 and Y_1 at some of these points. A table of 5-point Lagrange interpolation coefficients is also provided, for use with the tables. In the introduction are given contour plots for the general behavior of Y_0 and Y_1 in the complex plane; and a discussion of the properties of the functions and the method of calculation of the tables.

Together with the earlier volume,[1] giving corresponding values of J_0 and J_1 for complex values of the argument, these tables will allow many acoustical and microwave problems to be computed for complex boundary conditions on cylindrical surfaces. It is particularly satisfactory to have both functions J of the first kind and functions Y of the second kind, for the same values of the order and argument. Many applications of Bessel functions require knowledge of both functions and so many tables of these functions, heretofore published, have lost a great part of their value because only the J's were tabulated, not the Y's. In the present case the computing group is to be congratulated in resisting the temptation to compute J_2, J_3 etc., and in turning to the more difficult but more useful Y_0 and Y_1.

Perhaps some of our larger digital machines will eventually take the place of tables, making up in speed of output what they inevitably lack in judgment; they show little sign as yet of their ability to produce further tables, as carefully planned, edited, and produced as the tables under review. Perhaps, in the distant future, when we have a large digital computer in each laboratory, we will not need tables of the sort reviewed here (though the present reviewer thinks we always will). At any rate, for a long time to come, these tables and the many others produced by Lowan's able group will be standard and much used tools of physicists and engineers.

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[1] NBSCL, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments.* 2nd ed., New York, 1947 [*MTAC*, v. 3, p. 25].

913[L].—A. T. PRICE, "Electromagnetic induction in a semi-infinite conductor with a plane boundary," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 385–410.

Table I, p. 398, gives 3 and 4D values of Y_1, Y_2, Z_1, Z_2 for ξ, η = 0, .5, 1(1)5, 10 where

$$Y_n = \int_0^\infty f_n e^{-u^2} \cos u\eta \, du, \qquad Z_n = \int_0^\infty f_n e^{-u^2} \sin u\eta \, du,$$

$$f_1 = 1 - u\sqrt{2}\{(1 + u^4)^{\frac{1}{2}} - u^2\}^{\frac{1}{2}}, \qquad f_2 = 2u^2 - u\sqrt{2}\{(1 + u^4)^{\frac{1}{2}} + u^2\}^{\frac{1}{2}}.$$

A. E.

914[L].—P. RHODES, "Fermi-Dirac functions of integral order," R. Soc. London, *Proc.*, v. 204A, 1950, p. 396–405.

The functions

$$F_n(\eta) = \int_0^\infty x^n(e^{x-\eta} + 1)^{-1}\,dx$$

arise in the theoretical treatment of assemblies of particles subject to FERMI-DIRAC statistics. For the cases $n = \frac{1}{2}, \frac{3}{2}$ numerical tables are known.[1]

In the present paper n is a positive integer. A series convergent for negative η, a relation between $F_n(\eta)$ and $F_n(-\eta)$, and a polynomial approximation for large (positive) η are derived.

Table 1 (p. 404) gives the approximating polynomial for $n = 1(1)4$.

Table 2 (p. 404) gives 7D values of $F_n(\eta)/n!$ for $n = 1(1)4$, $\eta = -4(.1)0$.

A. E.

[1] J. MCDOUGALL & E. C. STONER, "The computation of Fermi-Dirac functions," R. Soc. London, *Phil. Trans.*, v. 237A, 1938, p. 67–104.
C. TRUESDELL, "On a function which occurs in the theory of the structure of polymers," *Annals Math.*, s. 2, v. 46, 1945, p. 150 [*MTAC*, v. 1, p. 445].

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **883** (VAN DER POL & SPEZIALI), **889** (DELURY), **894** (HARTLEY & PEARSON), **895** (HOWELL), **902** (AKADEMIÂ NAUK SSSR).

190.—A. J. C. CUNNINGHAM, *Binomial Factorisations*. V. 2, London, 1924.

Page 189, line 6
for 39249421 *read* 30249421 = 1291·23431

D. H. L.

191.—A. J. C. CUNNINGHAM & H. J. WOODALL, *Factorisation of* $(y^n \mp 1)$. London, 1925.

Page 17, $n = 66$, *delete* the factor 3
$n = 77$, *insert* the factor 463.

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192.—J. P. STANLEY & M. V. WILKES, *Table of the Reciprocal of the Gamma Function for Complex Argument*. Computation Centre, Univ. of Toronto, 1950 [*MTAC*, v. 5, p. 25–26].

The following errors occurred in the preparation for press. A number of illegible entries are also noted. Thanks are due to Miss C. M. Munford of the University Mathematical Laboratory, Cambridge, and to Dr. van Wijngaarden of the Mathematical Centre of Amsterdam, who helped in the discovery of these errors.

	Argument			
U	*x*	*y*	Column*	
	−0.17	0.18		*read* 179387
	−0.39	0.23		*read* 330349
	−0.38	0.31		*read* 380853
	+0.43	0.38		*for* 510159 *read* 519159
	+0.35	.40–.44	(arg)	*for* +.035 *read* +0.35
	+0.02	0.44	1	*read* 84522
	−0.41	0.45		*for* 525223 *read* 535223
	+0.43	0.45		*read* 535157
	+0.33	0.45		*for* 485888 *read* 385888
	+0.02	0.64	2	*for* 210157 *read* 201157
	+0.46	0.72	2	*for* 706845 *read* 706849
	+0.43	0.79		*for* 597722 *read* 697722
	−0.01	.84–.88	(arg)	*for* 0.01 *read* −0.01
	−0.45	0.85		*for* 1−.441265 *read* −1.441265
	0.00	0.92	1	*Insert* "−0"
	+0.04	0.97		*for* 393497 *read* 393947

	Argument			
V	*x*	*y*	Column*	
	−0.19	0.06		*read* 42836
	+0.33	0.06		*for* 70200 *read* 70290
	+0.42	0.16	1	*for* 185696 *read* 185686
	−0.45	.20–.24	(arg)	*for* −.045 *read* −0.45
	−0.35	.20–.24	(arg)	*for* −0.30 *read* −0.35
	−0.15	.32–.36	(arg)	*for* 0.15 *read* −0.15
	+0.50	0.45		*read* "+"
	+0.10	0.46		*read* "+"
	+0.05	0.47		*read* "+"
	+0.26	0.52	1	*for* 605490 *read* 695490
	−0.40	.56–.60	(arg)	*for* +0.40 *read* −0.40
	−0.31	0.64	2	*for* 435506 *read* 433506
	−0.09	0.71		*for* 844477 *read* 884477
	−0.40	.72–.76	(arg)	*for* 0.40 *read* −0.40
	−0.35	.72–.76	(arg)	*for* 0.35 *read* −0.35
	−0.25	.72–.76	(arg)	*for* 0.25 *read* −0.25
	−0.45	0.80	2	*for* +1.278102 *read* +0.278102
	+0.34	0.88	1	*for* 1422458 *read* 1442458
	−0.33	0.90		*read* 807496
	−0.34	0.92	2	*read* 820268
	−0.15	0.93		*read* 1352942
	−0.27	0.95		*for* 1155247 *read* 1115247
	−0.31	0.99		*for* 1012513 *read* 1102513

* Some columns are repeated on two pages, "1" identifies the first repetition, "2" the second.

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UNPUBLISHED MATHEMATICAL TABLES

122[B].—H. E. SALZER, *Table of* $n!/x^{n+1}$. Manuscript in possession of the author, NBSCL.

This table gives, to within a unit in the 9th place, the values of the function $n!/x^{n+1}$ for $n = 0(1)10$ and $x = .1(.1)10$. The table was computed with the aid of PETERS & STEIN'S values of the powers of the reciprocals on p. 36–57 of the *Anhang* to J. PETERS, *Zehnstellige Logarithmentafel*, v. 1.

123[F].—A. GLODEN, *Factorization of* $2n^2 + 1$ *for* $n = 1(1)1000$. Typewritten manuscript of 11 leaves deposited in UMT FILE and Scientific Computing Service, London, and in possession of the author, 11 rue Jean Jaurès, Luxembourg.

A table extending to $n = 3000$ is contemplated by the author.

124[F].—A. GLODEN, *Tables of Quadratic Partitions* $p = a^2 + 3b^2$ *for* $100000 < p < 200000$. Typewritten manuscript of 18 leaves, deposited as in UMT **123**.

The argument p is a prime of the form $6m + 1$. The table gives a and b for each p with some 35 omissions which are promised in an addendum [for previous tables up to $p = 125000$ see RMT **883**].

125[F].—R. M. ROBINSON, *Tables of Integral Solutions of* $|y^3 - x^3| < x$ *for* $x < 10^6/9$. Tabulated from punched cards and deposited in the UMT FILE. Cards in possession of the author, University of California, Berkeley.

There are 5 tables. The first gives all solutions arranged according to x, of which there are 1242. Table II lists the 332 cases in which $y^3 = x^3$. Tables III is arranged in order of $|y^3 - x^3|$. Table IV is for $y^3 > x^3$ and Table V is for $y^3 < x^3$. Tables IV and V omit the trivial solutions $x = 4k^3 \pm 1$, $y = 8k^3 \pm 3k$, which are starred in Tables I and III.

126[L].—G. FLAKE & Y. L. LUKE, *Tables of the Function* $(1 - uZ)^{-1/u}$. Lithographed manuscript, 4 leaves, deposited in UMT FILE and available from Midwest Research Institute, 4049 Pennsylvania Ave., Kansas City, Mo.

The table gives 8D values of this function for

$$z = .7(.01)1.25, \qquad u = .09(.01).12.$$

127[L].—C. P. GREEN, J. H. LILLIE & D. W. RAYNOLDS, *Extensive Tables of the Exponential Integral*. A thesis, Chemical Engineering Dept., Univ. of Tennessee. Knoxville, 1951, XXV + 111 leaves.

The main table is of $-\mathrm{Ei}(-x)$ for $x = 15(.0001)16$. There are also tables of $\mathrm{Ei}(x)$ and $-\mathrm{Ei}(-x)$ for $x = 15(1)50$ and interpolation coefficients as introduced by COULSON & DUNCANSON [*Phil. Mag.*, s. 7, v. 33, 1942, p. 745–760].

128[L].—G. JONES & D. UFFORD, *Table of the Functions* $C(P) = K_1(P)/[K_1(P) + K_0(P)]$ *and of* $PC'(P)$. Lithographed manuscript, 4 leaves, available as in UMT 126.

K_1 and K_0 are the usual Bessel functions. The tables give $C(P)$ and $-PC'(P)$ to 7D for $P = 0(.002).1(.01).3(.02)1(.1)2(.5)10(10)100$, and also for $P = .35(.05).95$.

129[L].—Y. L. LUKE & D. UFFORD, *Tables of the Function* $\bar{K}_0(x) = \int_0^x K_0(t)\,dt$. Lithographed manuscript, 3 leaves, deposited as in UMT 126.

The table gives 8D values of $\bar{K}_0(x)$ and of the auxiliary functions $A_1(x)$ and $A_2(x)$ defined by

$$K_0(x) = (\ln 2 - \gamma - \ln x)A_1(x) + A_2(x)$$

for $x = 0(.01).5(.05)1$.

130[L].—UNIVERSITY OF TORONTO COMPUTATION CENTRE, *Tables of Spherical Bessel Functions for Semi-imaginary Argument.* Photo copy, 2 leaves deposited in UMT FILE.

The tables give 8S values of the real and imaginary parts, absolute values and arguments of

$$(2x/\pi)^{-\frac{1}{2}}e^{-\pi i/4}J_{n+\frac{1}{2}}(xe^{\pi i/2})$$
$$(2x/\pi)^{-\frac{1}{2}}e^{-\pi i/4}Y_{n+\frac{1}{2}}(xe^{\pi i/2})$$

for $n = 0, 1, 2, 3$; $x = 0(1)10$.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 415 South Building, National Bureau of Standards, Washington 25, D. C.

Notes on Numerical Analysis—5

Table-Making for Large Arguments. The Exponential Integral

The evaluation of a function defined by a definite integral, for the complete range of argument $-\infty$ to $+\infty$, is usually performed in several stages. For small and moderate values of the argument x the integral is evaluated by means of an ascending series in powers of x, or perhaps by numerical quadrature. For very large values of x, numerical values are obtained by means of an asymptotic series.

The exponential integral, for example, defined by the equations

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} t^{-1}e^{t}\,dt$$

(1)

$$-\mathrm{Ei}(-x) = \int_{x}^{\infty} t^{-1}e^{-t}\,dt$$

has been tabulated by the use of ascending series,

$$\text{Ei}(x) = \gamma + \log x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

(2)

$$-\text{Ei}(-x) = -\gamma - \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!}$$

for a range of x up to about 15, and from the asymptotic series

$$\text{Ei}(x) \sim x^{-1} e^x (1 + 1!x^{-1} + 2!x^{-2} + \cdots)$$

(3)

$$-\text{Ei}(-x) \sim x^{-1} e^{-x} (1 - 1!x^{-1} + 2!x^{-2} - \cdots)$$

for larger values of x.

In some cases solutions for moderate values of x are obtained by numerical solution of the differential equation satisfied by the function, though this method does not appear to have been used for the exponential integral.

The use of the ascending series is rather cumbersome for $|x| > 10$, and the asymptotic series may not, without further refinement by the "converging factor" method of AIREY,[1] provide the required number of figures. This note gives a method for extending the tabulation, without any use of asymptotic series, to cover the whole range of x from $-\infty$ to $+\infty$. The method is not restricted to the exponential integral, but the latter provides a useful illustration.

The extension to large values of x is most easily performed by using an argument $z = 1/x$, and it is easy to show that the auxiliary function T, defined by the relation

$$T = e^{-z}\text{Ei}(x) - x^{-1} = e^{-1/z}\text{Ei}(z^{-1}) - z \tag{4}$$

satisfies the differential equation

$$T'' - (z^{-4} - 2z^{-3})T + z^{-2} = 0 \tag{5}$$

where dashes denote differentiations with respect to $z = 1/x$.

We can also show that the Taylor series at $z = 0$ has the form

$$T = z^2 + 2!z^3 + \cdots + n!z^{n+1} + \cdots \tag{6}$$

and is divergent for all $z > 0$. In spite of this fact we can use finite-difference equations and numerical integration to solve equation (5).

We assume that values of T are available, from previous computation with the ascending series, at $z = \pm 0.1$ ($x = \pm 10$), and we shall fill in the values for $z = -0.1(.01) + 0.1$ by the use of relaxation methods, producing by this means a table from which both $\text{Ei}(x)$ and $-\text{Ei}(-x)$ can be quickly obtained from the relation

$$\text{Ei}(1/z) = e^{1/z}(T + z) \tag{7}$$

For numerical purposes equation (5) is replaced by a set of difference equations to be satisfied at every pivotal point in the range of integration and typified by

$$T_1 + T_{-1} - \{2 + h^2(z_0^{-4} - 2z_0^{-3})\}T_0 + h^2 z_0^{-2} + \Delta(T_0) = 0$$

(8)

$$\Delta(T_0) = (-\delta^4/12 + \delta^6/90 - \cdots)\, T_0$$

At $z = 0$, T is zero and no further equation is needed, though it is easy to show from (6) and (8) that the final solution should satisfy the equation

$$T_1 + T_{-1} - 2h^2 + \Delta(T_0) = 0 \text{ at } z = 0.$$

The set of equations (8) is ideal for relaxation or any iterative process, as can be seen from the numerical values of the coefficients of T_0, for $h = 0.01$, which are given in the following table:

z	coeff. of T_0	z	coeff. of T_0
.01	9802	−.01	10202
.02	602	−.02	652
.03	118 + 4/81	−.03	132 + 70/81
.04	37 + 15/16	−.04	44 + 3/16
.05	16 + 2/5	−.05	19 + 3/5
.06	8 + 64/81	−.06	10 + 52/81
.07	5 + 1397/2401	−.07	6 + 1796/2401
.08	4 + 13/256	−.08	4 + 213/256
.09	3 + 1639/6561	−.09	3 + 5239/6561

The relaxation process follows familiar lines. A first approximation $T^{(0)}$ is obtained by neglecting Δ, an approximation to which is computed from the differences of $T^{(0)}$. The "missing" differences near the ends of the range are filled in by the methods of the previous note. (See *MTAC*, v. 5, p. 92–95.)

Two applications of the iterative process, the first of which produced a maximum change in T of 200, the second of 15 in the ninth decimal place, gave the results shown in the table appended.

From these values we can obtain by subtabulation a table from which $\mathrm{Ei}(x)$ and $\mathrm{Ei}(-x)$ can be computed for any $|x| \geq 10$. The only other table known to exist in this range is that of COULSON & DUNCANSON,[2] in which interpolation is by no means trivial.

It is also interesting to consider other numerical methods. The powerful method VII of FOX & GOODWIN,[3] for example, replaces equations (8) by

$$\begin{aligned}(1 - h^2 f_1/12)T_1 + (1 - h^2 f_{-1}/12)T_{-1} - (2 + 5h^2 f_0/6)T_0 \\ + h^2(5g_0/6 + g_1/12 + g_{-1}/12) + \Delta(T_0) = 0 \qquad (9)\end{aligned}$$

$$\Delta(T_0) = \delta^6/240 - 13\delta^8/15120 + \cdots$$

where

$$f = z^{-4} - 2z^{-3}, \qquad g = z^{-3}.$$

Only two cycles of the iterative process are now required; but the coefficients of T_1 and T_{-1} are no longer unity, and heavier arithmetic is called for in the relaxation. Equations (9) have been applied as a check, however, and indicate that the last figure given is nowhere in error by more than one unit. For small values of z, furthermore, more significant figures could be obtained trivially, a result of the large coefficient of T_0 in the finite-difference equations.

We also considered the solution by step-by-step methods of the first order equation

$$T' - z^{-2}T + 1 = 0. \qquad (10)$$

Method II of Fox & Goodwin[3] produces the recurrence relation

$$\begin{aligned}(1 - \tfrac{1}{2}hz_1^{-2})T_1 = (1 + \tfrac{1}{2}hz_0^{-2})T_0 - h + \Delta = 0 \qquad (11)\\ \Delta = (-\delta^3/12 + \delta^5/120 - \delta^7/840 + \cdots)T_{\frac{1}{2}}.\end{aligned}$$

Starting with $T = 0$ at $x = 0$ and integrating outwards, we run into trouble associated with the fact that the coefficient of T_1 passes through zero. Other methods have similar disadvantages, and indeed the presence of the large coefficients of T_0 in equations (8) suggests immediately that step-by-step methods will be difficult, relaxation relatively easy.

If we apply step-by-step methods using Taylor series, we soon run into trouble in working outwards from $x = 0$, owing to rapid growth of the complementary function introduced by rounding-off errors. Since it is, however, possible to work inwards from $x = 0.1$, this method was used as a spot check.

An independent computation has been performed by T. VICKERS, who used asymptotic series and interpolation in existing tables to produce values of the function

$$S = xe^{-x}\mathrm{Ei}(x) = xT + 1$$

in our previous notation. The limitations of the asymptotic series make it difficult to obtain more than seven decimals in S, effectively one less than we obtained in T. Vickers points out, however, that the differences of S converge more rapidly than those of T; in particular linear interpolation in S can be performed with an error of less than 2 units in the fourth decimal place.

Tables of S and T, with facilities for interpolation, are given in Table 1.

L. FOX & J. C. P. MILLER

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TABLE 1

x^{-1}	$10^9\,T$	δ_m^2	γ^4	S	δ_m^2
−.10	8436666	123964	0	0.9156333	1992
−.09	6936157	129007	0	0.9229316	2107
−.08	5564725	134430	0	0.9304409	2238
−.07	4327803	140290	0	0.9381742	2379
−.06	3231260	146634	1	0.9461457	2533
−.05	2281454	153538	1	0.9543709	2711
−.04	1485301	161068	1	0.9628675	2904
−.03	850351	169333	1	0.9716550	3124
−.02	384890	178446	1	0.9807555	3375
−.01	98058	188554	1	0.9901942	3663
.00	0	199858	1	1.0000000	3994
.01	102063	212593	2	1.0102063	4383
.02	417046	227108	2	1.0208523	4848
.03	959551	243875	3	1.0319850	5414
.04	1746477	263623	4	1.0436619	6124
.05	2797795	287587	7	1.0559559	7080
.06	4137929	318250	10	1.0689655	8467
.07	5798138	358767	11	1.0828305	10387
.08	7819020	409516	6	1.0977378	12551
.09	10250440	465725	− 3	1.1138938	14372
.10	13147020	518852	−12	1.1314702	15237

Formulae

$$T = e^{-x}\,\mathrm{Ei}\,(x) - \frac{1}{x}$$

$$S = xe^{-x}\,\mathrm{Ei}\,(x)$$

Interpolation

$$f_n = (1-n)f_0 + nf_1 + E_0{}^3\delta_{m0}{}^2 + E_1{}^3\delta_{m1}{}^2 + M_0{}^4\gamma_0{}^4 + M_1{}^4\gamma_1{}^4$$

(See BAAS *Table II*, Auxiliary Table I, for definitions and values of E and M.)

EDITORIAL NOTE: It is easy to find the value of $T(-x)$ for x positive and large, by use of the Laguerre approximate quadrature. We have

$$T(-x) = e^x \mathrm{Ei}(-x) + x^{-1} = x^{-1} - \int_0^\infty e^{-t}(x+t)^{-1}\,dt.$$

Now, approximately,

$$\int_0^\infty (x+t)^{-1}e^{-t}\,dt = \sum_{i=1}^{n} \alpha_i^{(n)}[x_i^{(n)} + x]^{-1} \tag{1}$$

where the $x_i^{(n)}$ are the zeros of the Laguerre polynomial and the $\alpha_i^{(n)}$ are the corresponding Christoffel numbers. These have been tabulated by SALZER & ZUCKER.[4]

The error in the formula (1) above[5] is the value of

$$E_n = \frac{(n!)^2 d^{2n}}{(2n)!\,dt^{2n}}(x+t)^{-1} = (n!)^2(x+t)^{-2n-1}$$

for some t, $0 < t < \infty$. This error is certainly less than

$$E_n = (n!)^2/x^{2n+1}$$

and since

$$E_{n+1}/E_n = (n+1)^2x^{-2}$$

the best n to use is $n = [x]$ which gives

$$E_n \sim 2\pi e^{-2n}$$

which is about 10^{-8} in the case $x = 10$. The accuracy obtained is therefore comparable with that of the above table for $z = .1$ and $z = .09$ and better when $z \geq .08$.

This method will be as convenient in the evaluation of functions of the same general form as $\mathrm{Ei}(-x)$ even when the differential equation which they satisfy is complicated, or awkward to obtain. It cannot, however, be applied conveniently to the case of negative x.

J. T.

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[1] J. R. AIREY, "The converging factor in asymptotic series and the calculation of Bessel, Laguerre and other functions," *Phil. Mag.*, v. 24, 1937, p. 521–552.

[2] A. C. COULSON & W. E. DUNCANSON, "Some new values for the exponential integral," Camb. Phil. Soc., *Proc.*, v. 33, 1942, p. 754–761.

[3] L. FOX & E. T. GOODWIN, "Some new methods for the numerical integration of ordinary differential equations," Camb. Phil. Soc., *Proc.*, v. 45, 1949, p. 373–388.

[4] H. E. SALZER & RUTH ZUCKER, "Table of the zeros and weight factors of the first fifteen Laguerre polynomials," Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 1004–1012.

[5] G. SZEGÖ, *Orthogonal Polynomials*. Amer. Math. Soc., *Coll. Publ.*, v. 23, 1939, p. 369.

BIBLIOGRAPHY Z–XVI

1. ANON., "Standards on electronic computers: definitions of terms, 1950," I.R.E., *Proc.*, v. 39, March 1951, p. 271–277.

2. EDMUND C. BERKELEY, "The uses of automatic computers in financial and accounting operations," *Journal of Accountancy*, v. 90, October 1950, p. 306–311.

The author feels that the needs of business in accounting are beginning to create a demand for the large electronic computers such as are now being

used in scientific and engineering problems. He explains, in this article, the fundamental nature of the large computer by comparing it in its essential characteristics with the ordinary desk calculator.

By analyzing the basic requirements of typical accounting problems, he is able to forecast the changes in computer design necessary to meet these requirements. The tangible benefits to business, such as the lower cost of the information produced and quick access to completely assimilated up-to-the-minute information, are forcibly pointed out. Of course, there will be construction and training delays, and it would be wise for business to study some of the problems suggested in the paper so that the machines can go to work as effectively as possible after their construction.

J. BLUM

NBSCL

3. ENGINEERING RESEARCH ASSOCIATES, Inc., *High Speed Computing Devices*. McGraw-Hill, 1950, 451 pages.

As stated in the preface, this book is primarily a discussion of the mechanical devices and electrical circuits which can be incorporated into computing machines. Included also are a brief treatment of the concepts of number theory which are pertinent to many electronic digital computing machines now in use or under development, a chapter on numerical analysis listing methods for solving algebraic and differential equations, and several chapters on computing systems.

The book contains three parts: Part I, The Basic Elements of Machine Computation; Part II, Computing Systems; and Part III, Physical Components and Methods. It appears to have been directed mainly at readers with some knowledge of engineering who wish to learn how large-scale digital computing machines function, and it serves this purpose well. The treatment of analogue devices is quite cursory, however, and the modicum of numerical analysis included serves best to justify the definition of component, given in the introduction, as "any physical mechanism or mathematical method which is used as a tool in automatic computation."

Granting the definition of component as given in the book, the classification of the material contained therein which is made in the introduction appeals to this reviewer. This classification of contents is the following:

A. General. Chapter 1, Introduction, Chapter 2, Preliminary Considerations; B. Mathematical Components. Chapter 6, Arithmetic Systems, Chapter 7, Numerical Analysis; C. Physical Components and Methods. Chapter 3, Counters as Elementary Components, Chapter 4, Switches and Gates, Chapter 13, Arithmetic Elements, Chapter 14, Transfer Mediums, Chapter 15, Data Conversion Equipment, Chapter 16, Special Techniques and Equipment for Possible Use in Computing Systems, Chapter 17, Factors Affecting Choice of Equipment; D. Computing Systems. Chapter 5, A Functional Approach to Machine Design, Chapter 8, Desk Calculators, Chapter 9, Punched-card Computing Systems, Chapter 10, Large-scale Digital Computing Systems, Chapter 11, Analog Computing Systems, Chapter 12, The Form of a Digital Computer.

As the chapter titles indicate, the book by the staff of the Engineering Research Associates touches but lightly on control and input-output elements of high-speed computing devices. By way of compensation, however,

the arithmetic element of such devices is treated in detail and Chapter 14 on transfer mediums, the largest chapter in the book, provides an excellent description of the physical phenomena basic to the high-speed storage elements in use in large-scale electronic computing machines. To those who are concerned most with electronic computing devices, Chapter 14 is probably the most interesting chapter of the book, covering both storage elements of the dynamic type, like the acoustical delay line and the magnetic drum, and electrostatic storage elements depending, for their action, on the secondary-emission phenomenon. The book, prepared by competent staff members of a company active in the development of large-scale, high-speed computers, has real merit in more than a single chapter, however. In the opinion of this reviewer, it is a valuable addition to the literature on computing devices, which will be most useful to those interested in large-scale electronic digital computers.

E. W. C.

4. S. GILL, "A process for the step-by-step integration of differential equations in an automatic digital computing machine," Camb. Phil. Soc., *Proc.*, v. 47, 1950, p. 96–108.

As the author observes, many of the methods of integrating differential equations which are effective for hand computation have serious disadvantages when the integration is to be performed on a high-speed automatic machine. Among the disadvantages may be listed:

(1) Starting values may be obtained by a process different from that emphasized in the remainder of the integration. The extra orders required to do this on a machine are wasteful of memory space.

(2) Preceding functional values may be required for the calculation at each point. The transferring of values and the "searching" which this necessitates is cumbersome and expensive of orders and time.

(3) Changing the interval may be complicated. Here again, the number of orders required to effect the change may be excessive.

These problems are particularly vexing when we are seeking to solve a large system of differential equations. The author, by an application of processes developed by KUTTA, has worked out an extremely effective method, having none of the above disadvantages, by which such systems can be solved. It is straightforward, easily coded and economical of storage space. The paper includes a fairly thorough discussion of the accuracy obtainable in such a scheme.

M. MONTALBANO

NBSCL

5. D. R. HARTREE, "Automatic calculating machines," *Math. Gazette*, v. 34, 1950, p. 241–252, illustrs.

This paper is the text of an address given before the Mathematical Association. Because this discussion was given before a mathematical group, the treatment of automatic calculators in general was limited. The author does touch upon some of the more important aspects of automatic machines with special reference to BABBAGE's dream of an "Analytical Engine," and the first realization of that dream, the Harvard Mark I Calculator. A short

discussion of the first machine using electronic circuits, the ENIAC, with illustrations of the main parts, completes the first part of the paper.

The second part is concerned with a discussion of the EDSAC, which is a serial binary machine with a storage system using ultrasonic acoustic delay units. This machine uses one-address code instructions and serial storage. It has one storage unit for current instructions and another for registering the address from which the current instruction is taken. Normally the contents of the latter are increased by unity for every instruction carried out. Numbers and instructions are received by the machine in coded decimal form by means of teleprinter tape, and output is on a teletypewriter. The numbers are operated on in the machine in the binary system.

The author concludes with a note on programming and coding and gives an example of coding for the EDSAC.

In this reviewer's opinion, the closing comment of the author is worthy of quoting. "These machines have been called 'electronic brains,' which carries the suggestion that they can 'think for themselves,' which they cannot do: they can only carry out, quite blindly, the sequence of instructions which has been thought out for them."

B. F. HANDY, Jr.

NBSCL

6. MARSHALL KINCAID, JOHN M. ALDEN, & ROBERT B. HANNA, "Static magnetic memory for low-cost computers," *Electronics*, v. 24, 1950, p. 108–111.

This paper reviews the basic theory and gives several applications for the static magnetic storage units using the special magnetic material Deltamax. The authors feel that these magnetic storage units have application in medium-speed, low-cost computers which could handle many problems encountered in daily operation of business and industry. The applications cited in the paper for use of these devices in computers are as follows:

1. Storage of information. The handling rate varies from zero to 30,000 pulses per second, and the information is not lost in the event of power failure.
2. Transformation of information. The devices transform information slow-speed to high-speed pulse systems and vice versa; they also change parallel-type information to serial-type information and vice versa.
3. A high-speed counter which can count up to 25,000 pulses per second using only simple auxiliary circuitry.

The paper concludes by mentioning some of the production techniques and difficulties encountered in the commercial production of these devices.

M. M. ANDREW

NBSMDL

7. B. L. MOORE, "Pentode counting or control ring," *Review Sci. Instr.*, v. 21, 1950, p. 337–338.

A counting or control ring circuit using 6AS6 pentodes in a very stable manner is described. The input signal is said to be essentially independent of amplitude and wave shape, and the values of circuit components and supply voltage are not critical. Control voltages may be taken from one or

more stable states of the counter, the maximum number being limited essentially by the power required to drive the grid. Combinations of counters may be used to obtain a ring of many stable states.

M. M. ANDREW

NBSMDL

8. ROBERT F. SHAW, "Arithmetic operations in a binary computer," *Review Sci. Instr.*, v. 21, 1950, p. 687–693.

The author gives a summary of the methods used to facilitate the basic arithmetic computations in contemporary binary computers. The use of complements is explained in defining negative numbers for machine use; corrections for multiplication error and round off procedures are discussed. With numerical examples, simply written but thorough, this paper is a good introduction to the logical mechanism of the binary computer.

KARL GOLDBERG

NBSCL

9. M. V. WILKES, "Automatic Computing," *Nature*, v. 166, December 2, 1950, p. 942–944.

Describes a Summer school course in coding for an automatic digital computing machine held at Cambridge in the University Mathematical Laboratory from September 12th through 21st. The course was concerned mainly with the methods used in connection with the EDSAC.

NEWS

Conference on automatic computing machinery and applications.—The conference was jointly sponsored by the Association for Computing Machinery, the Industrial Mathematics Society, and the Advisory Committee for Wayne University Computation Laboratory. There were general sessions on Tuesday morning and evening, March 27th, and two parallel groups of sectional meetings beginning Tuesday afternoon and continuing through Wednesday.

As the conference was held in the industrial center of Detroit, there was considerable interest in the applicability of computing machines to industrial problems. An exhibit of computer equipment was shown Monday evening through Wednesday afternoon. The program for the meeting was as follows:

General Session, Tuesday, March 27	FRANZ L. ALT, President, ACM, *Chairman*
Welcoming address	VICTOR A. RAPPORT, Dean, Wayne University
"Some computing problems in the automotive industry"	PAUL T. NIMS, Chrysler Corporation, Vice-president, Industrial Mathematics Society
"Automatic calculations and their applications"	HOWARD H. AIKEN, Director, Computation Laboratory, Harvard University
Sectional Meeting, Tuesday, March 27, Section A	ALEX L. HAYNES, Ford Motor Co., *Chairman*
"Digital computer research at M.I.T."	ROBERT R. EVERETT, M.I.T.
"The ENIAC—a five year operating survey"	W. BARKLEY FRITZ, BRL, Aberdeen Proving Ground
"Problems solved on B.T.L. Model VI Computer"	ERNEST G. ANDREWS, Bell Telephone Laboratories
"Capabilities and limitations of existing electronic equipment"	IDA R. RHODES, NBS
Sectional Meeting, Tuesday, March 27, Section B	C. C. HURD, IBM, *Chairman*

"Floating decimal calculations on the card programmed electric calculator"	RICHARD H. STARK, Los Alamos Scientific Lab.
"The treatment of systems of differential equations on the SEAC"	JOSEPH H. LEVIN, NBS
"Solution of the non-linear supersonic flow equations on the SEAC"	ETHEL C. MARDEN, NBS, J. CONRAD CROWN, NOL
"Preparation of a theodolite problem for large-scale machines"	BERNARD DIMSDALE, BRL, Aberdeen Proving Ground
Evening Session Dinner, Tuesday, March 27	JOHN R. RICHARDS, Wayne University, Toastmaster
"Electronics for business—luxury or necessity?"	JOHN S. COLEMAN, President, Burroughs Adding Machine Co.
Sectional Meeting, Wednesday Morning, March 28, Section A	ROBERT SCHILLING, General Motors Corporation, President, Industrial Mathematics Society
"The programming of some matrix computations for the Mark II Aiken Relay Calculator"	ALLEN V. HERSHEY, Naval Proving Ground
"The operation of the Fairchild specialized digital computers for certain applications in matrix algebra"	J. J. STONE, Fairchild Engineering and Airplane Corporation
"Matrix algebra programs for the UNIVAC"	H. F. MITCHELL, Jr., Eckert-Mauchly Computer Corporation (Subsidiary, Remington Rand Corp.)
"A UNIVAC program for inventory requirements"	G. M. HOPPER, Eckert-Mauchly Computer Corp. Subsidiary, Remington Rand Corp.)
Sectional Meeting, Wednesday Morning, March 28, Section B	ALBERT C. HALL, Bendix Aviation Corporation, *Chairman*
"Automatic checking features of the Raytheon Digital Computer"	LOUIS FEIN, Raytheon Manufacturing Company
"The external memory of the Raytheon Digital Computer"	KENNETH M. REHLER, Raytheon Manufacturing Company
"The digital reader"	GILBERT W. KING, Arthur D. Little, Inc.
"Preliminary considerations on a magnetic drum controlled computer"	THEODORE SHAPIN, Jr., E. F. MOORE, University of Illinois
"Techniques in the design of digital computers"	R. E. SPRAGUE, Computer Research Corporation
Sectional Meeting, Wednesday Afternoon, March 28, Section A	C. C. BRAMBLE, Naval Proving Ground, *Chairman*
"On the accuracy of Runge-Kutta's method"	MAX LOTKIN, BRL, Aberdeen Proving Ground
"Optimum trajectories"	ARNOLD S. MENGEL, The RAND Corporation
"Reversing digit number system"	GEORGE W. PATTERSON, Burroughs Adding Machine Company
"Analysis of digital computers in control systems"	W. K. LINVILL, J. M. SALZER, M.I.T.
Sectional Meeting, Wednesday Afternoon, March 28, Section B	EDMUND C. BERKELEY, Secretary, Association for Computing Machinery
"The impact of business on computing machinery design"	E. F. COOLEY, Prudential Insurance Company of America
"Plans for tabulating parts of the 1950 population census on an electronic computer"	JAMES L. MCPHERSON, Bureau of the Census
"Engineering applications on the MADDIDA"	GLENN E. HAGEN, Northrop Aircraft, Inc.
"Mathematical techniques in the uses of MADDIDA"	MYRON J. MENDELSON, Northrop Aircraft, Inc.

Institute on Computing Methods and Machines.—On January 26th through 30th inclusive a duplicate meeting was held at the University of California at Berkeley (Jan. 26–27) and at Los Angeles (Jan. 29–30). The purpose of the meeting was to provide a general picture of computer application to problem solutions and a specific picture of the computer facilities available in the University of California. The meeting was co-sponsored by various departments of the universities and by the NBSINA. The following program was presented:

Program	Berkeley	Los Angeles
Morning Session (Jan. 26 at Berkeley and Jan. 29 at Los Angeles)		
Chairman	H. A. SCHADE, U. C., Berkeley	L. CHERRY, NAMTC, Point Mugu, Calif.
Welcome and Introduction	M. P. O'BRIEN, Chairman, Dept. of Eng., U. C., Berkeley	L. M. K. BOELTER, Chairman, Dept. of Eng., UCLA
"Current status of digital computer developments"	E. U. CONDON, Director, NBS	
"The facilities and program of the Computer Laboratory at Berkeley"	PAUL L. MORTON, Director, Computer Lab., U. C., Berkeley	
"The California Digital Computer—CALDIC"	DAVID R. BROWN, U. C., Berkeley	
Afternoon Session		
Chairman	JERZY NEYMAN, Director, Statistical Lab. U. C., Berkeley	LOUIS A. PIPES, UCLA
"Role of the mathematics department in a university computing center"	DERRICK H. LEHMER, U. C., Berkeley	
"Computations relating to some solar system problems"	LELAND E. CUNNINGHAM, U. C., Berkeley	
"Facilities of the Institute for Numerical Analysis:		
a) problem formulating service	A. S. CAHN, NBSINA	
b) punched card equipment	E. C. YOWELL, NBSINA	
c) electronic equipment—SWAC"	HARRY D. HUSKEY, NBSINA	
Inspection of Computer Facilities		
Morning Session (Jan. 27 at Berkeley and Jan. 30 at Los Angeles)		
Chairman	ALFREDO BANOS, JR., UCLA	WILLIAM A. MERSMAN, Ames Aeronautical Lab.
"Recent developments in analog computer techniques"	W. W. SOROKA, U. C., Berkeley	
"The network thermal analyzers"	F. W. SCHOTT, UCLA	
"Electronic analog computations"	T. A. ROGERS, UCLA	
"Mechanical differential analyzer-problem solution"	DONALD LEBELL, UCLA	
Inspection of Computer Facilities		

The Institute of Radio Engineers.—At the National Convention of the Institute held in New York on March 19th through March 22nd, two sessions were devoted to computers. The programs were as follows:

Computers I	E. R. PIORE, Office of Naval Research, Washington, D. C., *Chairman*
"The Raytheon selection matrix for computer and switching applications"	KENNETH M. REHLER, Raytheon Manufacturing Company, Waltham, Mass.
"Saturable reactors as substitutes for electron tubes in high-speed digital computers"	JAMES G. MILES, Engineering Research Associates, Inc., St. Paul, Minn.
"Ferromagnetic cores for three-dimensional digital storage arrays"	WILLIAM N. PAPIAN, Servomechanisms Laboratory, MIT, Cambridge, Mass.
"Dependable small-scale digital computer"	J. J. CONNOLLY, Teleregister Corporation, New York, N. Y.
"An asynchronous control for a digital computer"	D. H. GRIDLEY, Naval Research Laboratories, Washington, D. C.
Computers II	J. W. FORRESTER, MIT, *Chairman*
"A sampling analogue computer"	JOHN BROOMALL and LEON RIEBMAN, Moore School of Electrical Engineering, Univ. of Penn.
"A time division multiplier for a general-purpose electronic differential analyzer"	R. V. BAUM and C. D. MORRILL, Goodyear Aircraft Corp., Akron, Ohio
"A high-speed product integrator"	ALAN B. MACNEE, University of Michigan, Ann Arbor, Michigan
"Plug-in units for digital computation"	G. GLINSKI and S. LAZECKI, Computing Devices of Canada, Ltd., Ottawa, Ontario, Canada
"A five-digit parallel coder tube"	J. V. HARRINGTON and K. N. WULFSBERG, Air Force, Cambridge Research Laboratories, Cambridge, Mass., and G. R. SPENCER, Philco Tube Laboratory, Lansdale, Pa.

National Physical Laboratory.—In November 1950 at the National Physical Laboratory a pilot model of the ACE (Automatic Computing Engine) was demonstrated to members of the British Press. The ACE is an electronic digital computer which is being constructed under the aegis of the Department of Scientific and Industrial Research by the staff of Electronics and Mathematics Divisions of the National Physical Laboratory at Teddington, Middlesex, in association with the English Electric Company of Stafford.

Owing to the size and complexity of the ACE itself, it was thought desirable to build first a smaller version known as the ACE pilot model. This is a serial machine operating on numbers in the binary scale, with a pulse repetition rate of one megacycle per second. It uses about 800 tubes in 40 plug-in units mounted on a rack about 12′ × 16′ and has separate control panel and power supply units. The internal memory is of the mercury delay line type and consists of eight long tanks, each with a delay of 1024 μsec. and 10 short tanks, each with a delay of 32 μsec. Storage capacity is 264 words, each word being the binary equivalent of a 10-decimal-digit number or an instruction in a three-address code. The time necessary to multiply two 10-decimal-digit numbers is .002 sec. For the input and output sections of the machine, modified Hollerith punched card equipment is used. The demonstration included the extraction of 3rd and 4th roots of 7-decimal-digit numbers, and a program for testing whether any given number less than a million is prime. Numbers, suggested by the audience, were fed into the machine in decimal form by hand switching. If the number was not prime, the lowest factor was indicated in decimal form by means of lights; if it was prime, after three seconds the machine sounded a buzzer. The machine was also used to solve a problem of practical importance—i.e., the tracing

of rays through a compound lens. The machine computed in a few minutes, the paths of 20 rays through a compound lens with six refracting surfaces. Such a computation would have taken a human computer two 7-hour days with a desk calculator.

Reeves Instrument Corporation.—Project Cyclone Symposium I on REAC Techniques was held in New York City, March 15–16, 1951 under the sponsorship of the Reeves Instrument Corporation with the approval of the U. S. Navy Special Devices Center.

It was the purpose of Project Cyclone Symposium I on REAC Techniques to provide a means for discussing techniques, applications and engineering difficulties which have arisen among users of REAC equipment, and to stimulate new techniques in the field of analogue computation.

The program consisted of five sessions, under the chairmanship of Stanley Fifer, head of Project Cyclone.

Morning Session, Thursday, March 15:

Welcoming addresses by David T. Bonner, President, Reeves Instrument Corporation, and Paul Staderman, Head, Synthetic Warfare Section, Special Devices Center.

1. "Optimum trajectories," Arnold S. Mengel, The RAND Corporation.
2. "Solution of some partial differential equations on the REAC," G. W. Evans II, Argonne National Laboratory.
3. "Analytical and mechanical methods of solution of differential equations with discontinuous forcing functions," William A. Mersman, National Advisory Committee for Aeronautics, Ames Aeronautical Laboratory.
4. "The solution of polynomial equations on the REAC," Louis Bauer and Stanley Fifer, Project Cyclone.
5. "Computation of multidimensional integrals on the REAC," Leslie C. Merrill, Argonne National Laboratory.

Afternoon Session, Thursday, March 15:

6. "RAND REAC modifications," Arnold S. Mengel and Wesley S. Melahn, The RAND Corporation.
7. "Some modifications for maximum utilization of the REAC," John L. Burnside, North American Aviation, Inc.
8. "The use of the REAC by Minneapolis-Honeywell in the analysis and synthesis of automatic flight control," Remus N. Bretoi and David L. Markusen, Minneapolis-Honeywell Regulator Company.
9. "Slaving of a rotational mount to an analog computer," C. F. Cook, National Bureau of Standards.
10. "Preliminary investigation of suitability of REAC for experimental curve fitting," Donald S. Teague, Jr. and R. D. Gilpin, U. S. Naval Air Missile Test Center, Point Mugu.
11. "The generation of an N dimensional normal distribution by means of analog equipment," Harry H. Goode, William A. Wheatley and George G. den Broeder, Willow Run Research Center, University of Michigan.

Morning Session, Friday, March 16:

12. "The generation and measurement of ultra low frequency random noise," Robert R. Bennett and A. S. Fulton, Hughes Aircraft Company.
13. "Simulation of noise," Harold L. Ehlers and Erwin Vogel, Fairchild Engine and Airplane Corporation.
14. "Frequency analysis by electronic analog methods," William A. McCool, Naval Research Laboratory.
15. "Application of REAC equipment to the solution of problems involving integral operators, I," Paul Brock and Seymour Sherman, Project Cyclone.

Noon Session, Friday, March 16:

Numerical checks; introduction by Paul Brock, Project Cyclone.

Afternoon Session, Friday, March 16:

16. "A REAC solution of a linear proton accelerator design problem," A. H. Miller, presented by Nathanial B. Nichols, University of Minnesota.
17. "Representation of functions of several variables on REAC equipment," Hans Meissinger, Project Cyclone.
18. "A method for solving problems on the REAC by the use of transfer functions without passive networks," Cyrus Beck, Naval Air Experimental Station, Philadelphia.
19. "REAC techniques," W. Frank Richmond, Jr., The Glenn L. Martin Company.
20. "The generation of straight line transfer relationships," Robert R. Bennett, Hughes Aircraft Company.

UNIVAC Acceptance Tests.—On March 23rd the first UNIVAC successfully passed its acceptance tests in the presence of representatives of the Bureau of the Census and the Bureau of Standards. A discussion of these tests will follow a brief description of this new computer.

The UNIVAC system consists of a computer with an elaborate Supervisory Control Console, one to ten Uniservos, one or more Unitypers, and one or more Uniprinters. There is also available a Punch-Card-To-Tape Converter, and a device to perform the reverse conversion is under construction. The computer is a decimal machine containing 1000 words in its 100 mercury delay lines. Each word consists of 12 alphanumeric digits and has room for two one-address commands. The pulse repetition rate of the machine is 2.25 megacycles allowing about 2000 additions or over 450 multiplications to be performed in one second. Each half of the twin Arithmetic Unit checks the operations of the other half and indicates on the Supervisory Console any discrepancy that may occur.

Each Uniservo uses an 8-channel metal tape with a magnetic plating. The tape is initially examined for flaws, which are marked and which the machine is constructed to skip over. These tapes move either forwards or backwards at ten feet per second, so that over 800 words can be delivered to the Memory or recorded from it every second. A reel of tape is over 1500 feet in length, has a capacity of 120,000 words, and may be reused some 700 times. It is completely erasable, moves in either direction, and can be rewound, all at the behest of the coder's commands.

The Unityper's keyboard possesses a number of keys in addition to the conventional alphanumeric characters. One set of these keys serves to prepare commands leading to the desired format in which the final results are to appear; another set calls into play the group of commands which appears on three tracks of the rudimentary memory built into the Unitypers. These commands can be set to facilitate, shorten, and correct the typing of the input information.

The Uniprinter converts the binary-coded information on the tapes into alphanumeric characters at the rate of 11 per second. At the same time it is able to carry out printing instructions stored on the tape such as: carriage return, upper case, ignore (i.e., do not print), etc. It should be added that a fast line printer is under construction which will list information at about twice the speed of the standard punch-card tabulator.

The Card-to-Tape Converter handles six 80-column cards per second, storing the contents of 10,000 cards on a reel of tape and stopping automatically after a preset number of cards have been converted.

The four acceptance tests which the UNIVAC system underwent successfully consisted of:

1. A general test with emphasis on the computer's performance. It was subdivided into three parts. The first, which was repeated 404 times per test, investigated the

reliability of the execution of every one of the computer's commands. The second part required the computer to sort a number of items obtained as a result of the operations in the first part, as well as to solve a rather complicated partial differential equation. In the last part, the tapes of the four Uniservos were run through a difficult routine of a series of movements, delivering and recording information. Two such tests, requiring twenty minutes for a flawless performance, were designated as a unit. Out of the successive nineteen units which the UNIVAC performed during the test period, sixteen were carried out perfectly; in the three units where stops occurred due to the action of the error-detection circuits, the operator was able to rectify them by the mere manipulation of the console buttons.

2. A Uniprinter test requiring a sentence and a numerical table to be printed out in proper form 200 times. This test lasted eight hours, during which five errors occurred, whereas the conditions of the test allowed eight.
3. A Card-to-Tape-Converter test, which was passed satisfactorily. Although required to process a deck of 10,000 very badly punched cards, the Converter made only one mistake while storing this information repeatedly on ten tape reels.
4. A general test, with emphasis on Uniservo performance. Various aspects of tape movements were under scrutiny such as any tendency of the tapes to move out of position during prolonged inactivity, the efficiency of flaw-detection in the magnetic coating of the tapes, the resistance of this coating to a continuous rerunning of the tape under the reading heads for 700 times, the effect that various juxtapositions of tape commands might have on their performance, and several other points of possible weakness. This test lasted ten hours during which six errors were detected by the machine.

The machine will remain on the premises of the Eckert-Mauchly Computer Corporation for about a year, performing computations for the Bureau of the Census and operating 24 hours a day.

OTHER AIDS TO COMPUTATION

Bibliography Z–XVI

10. B. P. Bogert, "A network to represent the inner ear," *Bell Lab. Record*, v. 28, 1950, p. 481–485.

An electrical network consisting of 175 sections is used to simulate the cochlea which is believed to be the frequency sensing portions of the ear. The results of measurements on this simulator are compared for a number of frequencies with the results computed from the theory developed in a paper by Peterson & Bogert.[1] The simplified model upon which a hydrodynamical theory is based consists of two parallel ducts separated by a movable membrane of varying width. Measurements on the electrical network are compared with the numerical solution of the equations of wave propagation in this model, in regard to the amplitude of the response of the membrane at different points along its length to different frequencies. The response patterns vary considerably with the frequency but good agreement was obtained between the network, the theoretical model and certain experimental results for the actual cochlea.

F. J. M.

[1] L. C. Peterson & B. P. Bogert, "A dynamical theory of the cochlea," Acoust. Soc. Amer., *Jn.*, v. 22, 1950, p. 369–381.

11. J. W. CLARK & R. E. NEUBER, "A dynamic electron trajectory tracer," I.R.E., *Proc.*, v. 38, 1950, p. 521–524.

In the gravitational analog of a vacuum tube, a surface reproduces the electric potential in the tube and electrons are simulated by metal balls. By vibrating this surface, the authors reproduce the effect of the application of radio frequency voltages to various electrodes. Relatively low mechanical frequencies may be used. The results are recorded by a stroboscopic photographic process. A number of such photographs are given as illustrations.

F. J. M.

12. H. L. CURTIS, "Determination of curvature by an osculometer," NBS, *Jn. Research*, v. 44, 1950, p. 131–134.

The radius of curvature of a curve is determined graphically by means of a series of pairs of arcs of known curvature drawn on transparent paper. The user tries to arrange a pair of arcs symmetrically on both sides of the given curve. "The curvature at a point can usually be determined to 10^{-3} reciprocal centimeters, provided the curvature is approximately constant for a distance of 2 cm. . . ." A discussion is given for the graphical determination of accelerations by this method.

F. J. M.

13. T. S. GRAY & H. B. FREY, "Acorn diode has logarithmic range of 10^9," *Rev. Sci. Inst.*, v. 22, 1951, p. 117–118.

The diode type 9004 with a d.c. filament voltage supply of 4 volts has a logarithmic current characteristic between 10^{-4} and 10^{-13} amperes for the plate voltage range of 0 to -1.75 volts.

F. J. M.

14. H. IAMS, "A method of simulating propagation problems," I.R.E., *Proc.*, v. 38, 1950, p. 543–545.

Propagation problems can be effectively simulated by propagating centimeter wavelength radio waves in large sheets of low loss dielectric materials. The conditions of the problem are simulated by imbedding objects in the sheet or utilizing the effect of varying the thickness of the sheet to simulate a change in refractive index. The propagation is recorded by means of a "phase front plotter" which records the lines of equal phase. A coupling set up to simulate waves propagated from a large distance is described.

F. J. M.

15. G. A. KORN, "Stabilization of simultaneous linear equation solvers," I.R.E., *Proc.*, v. 37, 1949, p. 1000–1002.

The author gives an experimental test involving only one equation on an amplifier such that stability in this test will permit the amplifier to be used in a linear equation solver of the Goldberg-Brown type for any number of unknowns. A discussion of this paper, by L. A. ZADEH and the author, will be found in I.R.E., *Proc.*, v. 38, 1950, p. 514.

F. J. M.

16. G. LIEBMANN, "Solution of partial differential equations with a resistance network analogue," *Brit. Jn. Appl. Phys.*, v. 1, 1950, p. 92–103.

The author describes a resistance network analogue which can be used for the solution of Laplace's equation. The author points out that such a network may be considered as a computing machine for carrying out the finite difference method. The expected accuracy of the resistance network solutions is influenced by the finite mesh size and by resistors which deviate from the theoretical value. The error caused by finite mesh size can be greatly reduced by an extrapolating method due to Richardson. This method is briefly discussed in the paper. An examination of the errors due to deviations in resistance values follows but in order to evaluate this error the author must assume random distribution of deviations from the theoretical value. With this assumption he finds that the errors are exceedingly small. However, since the distribution of resistance deviations cannot usually be predicted definitely, the method of the author does not permit the determination of the maximum error incurred with a network.

In describing the design of a network he presents a valuable method of overcoming the difficulty of transition from coarser to finer lumping (his Figure 11). The actual description of a network with which the author works shows that he has assembled more than 2600 resistors. The description is all too brief and gives the impression that the author has designed a network with fixed resistance values instead of using as frequently done resistors which could be set to desired values. Using fixed resistors would seriously limit the applicability of his apparatus. He uses alternating current as power supply and an oscilloscope for balance indicating. It would appear that direct current would be experimentally easier. The bibliography overlooks an important British publication which obviates the necessity in the case of a study of cylinders to replace a solid cylinder by a hollow one.[1] Moreover, he disregards the considerable literature on such networks used in this country. The Heat and Mass Flow Analyzer Laboratory at Columbia University has issued over the last 10 years some 50 papers all dealing with work on a resistance capacitance network which of course is a more inclusive system than the one dealt with in the paper under review. A reasonably complete bibliography of this laboratory up to 1947 is contained in reference.[2]

V. PASCHKIS

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[1] R. JACKSON, R. J. SARGANT, J. B. WAGSTAFF, N. R. EYRES, D. R. HARTREE & J. INGHAM, "Variable heat flow in steel," Iron & Steel Inst., *Jn.*, v. 150, 1944, No. 2, p. 211–268.

[2] V. PASCHKIS, "The heat and mass flow analyzer—tool for heat research," *Metal Progress*, v. 52, 1947, p. 813–818.

17. W. S. MCEWAN & S. SKOLNIK, "An analog computer for flame gas composition," *Rev. Sci. Inst.*, v. 22, 1951, p. 125–132.

A problem in chemical equilibrium reduces to the problem of solving a system of equations, some of which are linear in the unknowns, others are ratios which may involve square roots of the unknowns. The device described realizes the unknowns as resistances and the equations are set up

as Wheatstone bridge circuits. While ganged potentiometers are used in certain instances, a switching system permits one to realize various different equations with the same resistance as unknowns in most cases. The device is manipulated until all the bridge circuits are in equilibrium. The manipulation is guided by certain physical information.

F. J. M.

18. A. MANY, "An improved electrical network for determining the eigenvalues and eigenvectors of a real symmetric matrix," *Rev. Sci. Inst.*, v. 21, 1950, p. 972–974.

An electrical network for the determination of the characteristic roots and vectors of a symmetric matrix of order ten, consisting of ten LC-circuits coupled to each other by equal condensers, is described. (The characteristic roots correspond to the resonant frequencies of the LC-circuits.) A similar but smaller and less accurate machine was described by MANY & MEIBOOM.[1] In the present paper the effects of unequal self-inductance of the coils of the tank circuits is compensated for. The stray capacitance of the wiring of the network is also compensated for by trimming the coupling condensers. Two important factors affecting the accuracy of the eigenvalues are: (1) Accuracy with which the matrix is realized, and (2) Losses in the coils. The above discussion compensates for (1). Using some results of the earlier paper, it is pointed out that the accuracy for matrices with reasonably well separated roots is mainly determined by (1) while for matrices with close eigenvalues, (2) predominates. In the first case, the accuracy obtained was about 0.01% of the difference of the greatest and smallest roots.

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[1] A. MANY & S. MEIBOOM, "Electrical network for determining the eigenvalues and eigenvectors of a real symmetric matrix," *Rev. Sci. Inst.*, v. 18, 1947, p. 831–836.

19. R. W. MARSHALL, "An integraph for semicurvilinear coordinate paper," *Bell Lab. Record*, v. 28, 1950, p. 50–52, 65.

When a function $y(t)$ is obtained from the output of a customary recording instrument, it is plotted on curvilinear paper, i.e., the output is measured along an arc of a circle of fixed radius. The device described in this article has an input arrangement which changes this displacement to a linear displacement. This linear displacement is fed into a CORADI integraph which plots a curve corresponding to the integral of the original function.

F. J. M.

20. G. E. REYNOLDS, *A New Method for Square Root Evaluation, Particularly to Eight Significant Figures with Monromatic Calculator*. Publication E4077 Unclassified. 22 p. Air Force Cambridge Research Laboratories, Cambridge, Mass., 1951. 21.4 × 27.6 cm.

The booklet's introduction states that it stems from ideas presented by R. C. Spencer in unpublished memoranda of 1948, and also that with some changes the method is adaptable to the Fridén and Marchant calculators, as outlined in an appendix.

By use of a compact optimum-interval table of certain squares of selected numbers as well as of 4-times the numbers (in an adjacent column), a root with error not exceeding 5 in 9th place is obtained by comparatively simple calculator work. This results from combining two Newton-type iterations to a square root; i.e., the mean of quotient and divisor after dividing a number whose root is desired N by an approximate root A obtains a root substantially correct to twice the number of digits that are correct in the first approximation of the root. This is the first time so far as the reviewer knows that two successive Newton iterations have been combined for calculator usage as a continuous process. Two divisions are required, as is to be expected.

The optimum-interval table used has 138 lines of which the following are typical:

APPROX.	DIV.
11.8336	13.76
12.2500	14.00
12.6736	14.24

The left-hand number is the square of a first approximation of $N^{\frac{1}{2}}$, or A^2, and the right-hand number is $4A$, which are applied as follows:

$$(N + A^2)/4A + 0.25N/((N + A^2)/4A) = N^{\frac{1}{2}} + \epsilon$$

The selection of A^2 and $4A$ values for the table is so made that ϵ does not exceed 5 in 9th place of $N^{\frac{1}{2}}$.

The approximation amount A^2 may be either the tabular value above or below N when the latter is pointed off in the manner of the tabular values.

The method of obtaining the tabular intervals so as to minimize the size of the table is given, as are curves from which the intervals may be determined for preparation of tables for obtaining 7-, 8-, or 9-figure roots, though only the table for 8-figure roots is given. A feature of this part of the explanation is that a complicated expression involving the interval between tabular values and the approximate root for various table ranges reduces to an exceedingly simple form. The final formula used for calculator work, as above, is notable because the quotient of the first division becomes the divisor of the second. In the case of the Monromatic Calculator, a single entry of N suffices for both terms, it being held in the storage register for multiplication by 0.25 to obtain the numerator of the second term.

The appendix states that the method is applicable to late-model Fridéns without change of formula, but a change with an additional step is recommended for the Marchant. The reviewer submits that a change is not required in this instance because even though the first quotient remains in the register so the second quotient may be accumulated, the keyboard entry of N for starting the second term may still be multiplied by 0.25 to obtain $0.25N$ in the product register, provided N is then cleared from the keyboard and a negative entry of 0.25 is made in the multiplier keyboard to restore the original quotient. This is readily done on any Marchant having automatic multiplication.

The proposed machine method requires that certain exceptions have to be made to the general rule that the first significant figure of all entries is to be made at the extreme left of keyboard; no decimal markers apparently

are used. The reviewer suggests that unless extreme-left entries can be made without exception, it is better practice and requires less attention by the operator if there is a suitable decimal setup, and all entries are made with reference to it. As the table shows its amounts pointed off in the range 1–100, any N may similarly be pointed off without loss of generality; i.e., 82547.355 would be considered as 8.2547355. A suitable decimal setup then would be: counting register, at left of 8; product register, at left of 16; and keyboard, at left of 8; set No. 9 tab key. This setting accommodates at least 9-figure N's and roots to 9 digits as well as intermediate amounts during the solution; an 11th dial in the counting register is not required.

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21. R. Roscoe, "Mechanical models for the representation of visco-elastic properties," *Brit. Jn. Appl. Phys.*, v. 1, 1950, p. 171–173.

The author establishes the existence of two canonical forms for the networks of springs and dashpots used to represent visco-elastic properties. One form is parallel, the other is essentially a series arrangement of parallel pairs of springs and dashboards with a simple series addition. The author's discussion is based on the analogy with resistor-capacity circuits for which the corresponding result is known. (Cf. Guillemin, "Communication Networks," v. 2, Chap. V, p. 184–221, John Wiley and Son, New York, 1947.)

F. J. M.

22. L. H. Wilson & A. J. Miles, "Application of the membrane analogy to the solution of heat induction problems," *Jn. Appl. Phys.*, v. 21, 1950, p. 532–535.

The authors describe the application of the membrane analogy to the solution of heat conduction problems. Mathematical justification is obvious from the identity of the equations and is dealt with only briefly. In order to carry out the technique, a membrane is created by using a soap film. The deviation of distance of various parts of the membrane from a reference point is proportional to the deviation of temperature in an equally shaped plane from the temperature of the corresponding reference point. The membrane is stretched across appropriate boundaries; for example, if heat flow through a hollow cylinder is to be studied, the two surfaces of which are at different temperatures, an annular membrane must be stretched between two cylindrical bodies serving as boundaries and placed at different levels according to the different temperatures of the outside and inside surface of the cylinder. The authors show two examples of comparison of determinations carried out by this method against theoretical curves. In one case the check is very good and seems to be better than 1%. The other example shows deviations which are, in part, as large as 10%.

Whereas the technique undoubtedly has some merits, the following serious limitations are not discussed and not even mentioned in the paper. First, in many thermal problems, the surface temperatures are not given; instead the temperature of the ambient is known and a boundary conductance applies, which may or may not be introduced as a function of the surface

temperature. Application of such boundary conductances seems impossible with the present technique. Secondly, in many steady-state problems two or more materials of different conductivity enter the picture. Application of the membrane technique to such cases does not seem possible. The authors mention the desirability of investigating the membrane analogy in the realm of transient-heat conduction problems. There is no indication either on a mathematical or experimental ground that such approach is possible.

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NOTES

127. AN ITERATIVE PROCESS.—In a recent paper HARTREE[1] calls attention to the need for the development of a general method of obtaining iteration formulas of second and higher orders for the solution of any "algebraic" (i.e., not differential) equation. If $y = Y$ is the true solution of $F(y) = 0$, and $y_n = Y + \eta_n$ is an approximation to Y, the formula $y_{n+1} = G(y_n)$ is an iterative process. If $\eta_{n+1} = O(\eta_n{}^r)$, Hartree calls the process r-th order. The number of correct figures in y_{n+1} is approximately r times the number in y_n, so it is obvious why a high-order process is desired.

The NEWTON-RAPHSON process, based on

$$G(y) = y - F(y)/F'(y)$$

is second order, as Hartree points out. In effect, this process draws a tangent to the curve $Z = F(y)$ at $y = y_n$, and takes y_{n+1} at the intersection of this tangent with the y-axis. The primary source of error is the curvature of F, and any operation that reduces the curvature will improve the convergence of the iteration. If we write

$$H(y) = F(y)\{F'(y)\}^{-\frac{1}{2}}$$

we find

$$H''(y) = -\tfrac{1}{2}F(y)\{F_0''(y)(F'(y))^{-3/2}\}'.$$

The function $H(y)$ consequently has the same roots as F, and zero curvature at each of them. If we apply the Newton-Raphson process to it we get

$$G(y) = y - 2FF'/(2F'^2 - FF''),$$

and it is easily verified that this process is third-order.

As an example, if $F(y) = y^2 - a$ we get

$$y_{n+1} = y_n(y_n{}^2 + 3a)/(3y_n{}^2 + a)$$

as a third-order process for computing $a^{\frac{1}{2}}$. For $a = 10$, $y_0 = 3$, we find $y_1 = 3.16216$, $y_2 = 3.162277660168341$, which is in error by 4 units in the fifteenth figure.

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[1] D. R. HARTREE, "Notes on iterative processes," Camb. Phil. Soc., *Proc.*, v. 45, 1949, p. 230.

QUERIES

38. GAUSS-SEIDEL PROCESS.—Is there any reason why Gauss' name should be associated with this method of solving linear systems?

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QUERIES—REPLIES

48. EXPONENTIAL INTEGRALS FOR COMPLEX ARGUMENT (Q 36, v. 5, p. 55).—See the tables by E. KREYSZIG reviewed in this issue [RMT 909].

CORRIGENDA

V. 4, p. 66, equation (2) *for* h *read* k.
V. 4, p. 70, equation (25) *for* $= B^k$ *read* $= \alpha B^k$.
V. 5, p. 3, l. 22, *for* (-1^N) *read* $(-1)^N$.
V. 5, p. 55, l. 5, *for* L *read* 1.
V. 5, p. 72, RMT 862, l. 4, for ∞ read $-\infty$.
V. 5, p. 73, RMT 865, l. 4, for x read $\bar{x}$; l. 6, for x, read x_1.